Analyzing Based on Combinatorics: Mahjong with Graph Theory, Generating Functions, and Computer-based Method

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Abstract

In this paper, we construct a graph model for mahjong to prove that the distance formula is correct. Also, we estimate the number of winning hands based on generating functions, and count the exact number using the brute-force method by computer. Moreover, by using the brute-force method, we check every possible type of ready hands.

1 Introduction

Back to the 1910s, the twilight of the Qing dynasty, there was a game played by people of China, which is known as had born in the mid-19th century. This game went overseas, reached the USA, and was widespread during 1920-1930s. It was spread enough to start a league for this game in 1937. The name of this game is *mahjong*.

Mahjong is a board game that depends much on luck, and this randomness restricts methods to analyze the game. In other words, mahjong is a game without perfect information. Moreover, mahjong has a complicated structure and innumerable variations based on more than 100 tiles, which makes it difficult to develop and generalize theory.

Hence, in this paper, Section 2 focuses on developing theory based on the graph theory to compute specific static data from mahjong, which can be applied to mahjong generally, and Section 3 focuses on various computed data from Japanese mahjong, which is called *rīchi mahjong*. For readers who are not familiar with mahjong, Appendix 1 explains rules and terms for mahjong, so it is recommended to read first.

2 Section 2

In this section, we consider graphs as undirected simple graphs. Also, for a relation $R \subseteq X \times Y$, we use the notation $R(a) = \{b \in Y \mid (a, b) \in R\}$ like functions. For a graph G, R_G is the binary relation on $V(G)$ such that $(u, v) \in R_G$ if and only if u and v are adjacent. Since we are considering undirected graphs, $R_G = R_G^{-1}$.

For a graph G, a path is a sequence of adjacent vertices without repetition and its length is defined as the number of edges consisting it. Note that a vertex itself is a path of length 0. A walk is a sequence of adjacent vertices in G, where repetition is allowed. With mathematical induction, it is easy to prove that for nonempty subsets S, T of vertices, if there is a walk connecting S and T, then there is a path connecting S and T, which is not longer than the original walk. Now, for a graph G and nonempty subsets S, T of vertices, define the distance $d_G(S, T)$ as the length of a shortest path connecting S and T. From the above fact, this definition is equivalent to the length of a shortest walk.

Naturally, we are considering undirected graphs, $d_G(S, T) = d_G(T, S)$. For any vertex $v \in V$, we will define $d_G(v, -)$ as $d_G(\{v\}, -)$. Lastly, if S and T are disconnected, then we note $d_G(S, T) = \infty$ as a convenction.

Proposition 1. For a graph G and a nonempty subset S of vertices, the following is satisfied.

(a) For any $v \in V$, $d_G(S, v) = 0$ if and only if $v \in S$.

(b) For any $u, v \in V$ such that u and v are adjacent, then $d_G(S, u), d_G(S, v) < \infty$ and $|d_G(S, u) - d_G(S, v)| \leq 1$, or $d_G(S, u) = d_G(S, v) = \infty$.

(c) For any $v \in V$, if $d_G(S, v) < \infty$ and $d_G(S, v) > 0$, then there exists $u \in R_G(v)$, in otherwords, adjacent to v, such that $d_G(S, u) < d_G(S, v)$.

Proof. At first, (a) follows from the definition. Now, for (b), we may assume u, v are connected to S. If P is a shortest path from S to u, then $S \xrightarrow{P} u \longrightarrow v$ is a walk from S to v with length $d_G(S, u) + 1$, so $d_G(S, u) \leq d_G(S, v) + 1$. By symmetry, $|d_G(S, u) - d_G(S, v)| \leq 1$.

For (c), if $v_0, v_1, \cdots, v_n = v$ is a shortest path from S to v, then v_{n-1} is such vertex. □

Moreover, these conditions also define the distance function in the following sense.

Proposition 2. For any graph G and function $d : V \to \mathbb{N} \cup \{0, \infty\}$ with $S \subseteq V(G)$, the following holds.

(a) If for $v \in V(G)$, $v \in S$ implies $d(v) = 0$ and $|d(u) - d(v)| \leq 1$ for any adjacent u and v, then $d(v) \leq d_G(S, v)$ for any $v \in V(G)$.

(b) Suppose for $v \in V(G)$, $d(v) = 0$ implies $v \in S$ and for any $v \in V$ such that $0 < d(v) < \infty$, there exists $u \in R_G(v)$ satisfying $d(u) < d(v)$. Then, $d_G(S, v) \leq d(v)$ for any $v \in V(G)$.

In particular, if function d satisfies conditions (a),(b),(c) of **Proposition 1**, then $d(v) = d_G(S, v)$.

Proof. (a) We may assume $d_G(S, v) < \infty$. If $d_G(S, v) = 0$, then $v \in S$, so $d(v) = 0$. Now, assume $d_G(S, v) = n > 0$. Suppose $v_0, \cdots, v_n$ is a shortest path from S to v. Then, $d(v_1) \leq d(v_0) + 1 = 1$ since $v_0 \in S$. Now, $d(v_2) \leq d(v_1) + 1 \leq 2$, and repeat this, we gain $d(v_n) \leq n = d_G(S, v_n) = d_G(S, v)$.

(b) We may assume $d(v) < \infty$. If $d(v) = 0$, then $v \in S$, so $d_G(S, v) = 0 = d(v)$. Now, if $d(v) = n > 0$, then let $v_0 = v$. Then, choose v_1 such that $v_1 \in R_G(v_0)$ and $d(v_1) < d(v_0)$, so $d(v_1) \leq n - 1$. By repeating this, we gain a path P, which is defined as $v_0, v_1, \cdots, v_k$ with $d(v_k) = 0$. This is a path since $d(v_i)$ are all distinct. Then, $d(v_i) \leq n - i$ by induction, so $k \leq n$, and hence, the length of P is at most n. Moreover, $d(v_k) = 0$, so $v_k \in S$. Thus, P is a path from S to v with length k, so $d_G(S, v) \leq k \leq n = d(v)$. $\square$

Definition 3. For two graphs G, G' and relation $R \subseteq V(G) \times V(G')$, $G//_R G'$ means $\{a\} \cup R_G(a) = R^{-1}(R(a))$ and $\{b\} \cup R_{G'}(b) = R(R^{-1}(b))$ for any $a \in V(G), b \in V(G')$.

Proposition 4. For two graphs G, G' and relation R such that satisfies $G//_R G'$, $d_G(S, R^{-1}(T)) = d_{G'}(R(S), T)$ for any $S \subseteq V(G), T \subseteq V(G')$. In particular, $d_G(R^{-1}(T), v) = d_{G'}(R(v), T) = \min_{u \in R(v)} d_{G'}(T, u)$ for any $v \in V(G)$.

Proof. Suppose $v_0, \cdots, v_n$ is a shortest path between S and $R^{-1}(T)$. Without loss of generality, assume $v_0 \in R^{-1}(T)$ and $v_n \in S$. Then, there exists $u_0 \in T$ such that $v_0 \in R^{-1}(u_0)$. Now, $v_1 \in R_G(v_0) \subseteq R^{-1}(R(v_0))$, so there exists $u_1 \in R(v_0)$ such that $v_1 \in R^{-1}(u_1)$. Now, $u_1 \in R(v_0) \subseteq R(R^{-1}(u_0)) = \{u_0\} \cup R_{G'}(u_0)$. By repeating this process, we define $u_0, \cdots, u_n$ such that $u_0 \in T$, $u_i \in u_{i-1} \cup R_{G'}(u_{i-1})$, $u_i \in R(v_{i-1})$, and $v_i \in R^{-1}(u_i)$. Then, $v_n \in R^{-1}(u_n)$, so $u_n \in R(v_n) \subseteq R(S)$. Now, if we consider $u_0, \cdots, u_n$, then each u_i is adjacent to u_{i-1} or $u_{i-1} = u_i$, so it gives a path from $u_0 \in T$ to $u_n \in R(S)$ with the length at most n. Thus, $d_{G'}(R(S), T) \leq d_G(S, R^{-1}(T))$. By symmetry, we get equality. $\square$

With these tools, we will estimate and prove the formula for the number of turns needed to be the ready hand from given hand, in abstracted mahjong. Abstracted means that, at first, we assume every tile has no restriction on the number. This absraction is reasonable since on the usual mahjong since every tile has the possibility of being a triplet, and we consider technical ready hands, which gives flexibility.

Second, we will assume without proof that the number of turns only related to the numbers of triplets, sequences, pairs, waiting for a sequence, and remaining tiles. This can be justified even though we have many possibilities in numbers of triplets, sequences, pairs, a step before sequence, and others, yet we can choose the minimum among them. For example, if the hand contains 11123, it can be considered as 11/123 or 111/23. If we consider it as 11/123, then it becomes one pair and one sequence, and if we consider it as 111/23, then it becomes one waiting for a sequence and one triplet. For any case, we will consider every possibility and get the minimum. Since then minimum among minimized value is again the minimum, this abstraction will work.

Lastly, we will assume that the number of triplets and sequences also does not have effect on getting the formula. Since for any mahjong, the goal is making sequences and triplets, so if we complete it, then there is no need to consider them again.

Then, each state of abstracted mahjong can be denoted as a triple (a, b, c) of nonnegative integers, where a is the number of pairs, b is the number of waiting for a sequence, and c is the number of remaining tiles. Then, the number of tiles in the hand of a non-playing player is $3k + 1$ for some integer k, and the number of tiles in the hand of the playing player is $3k + 2$. In otherwords, if non-playing, $2a + 2b + c = 3k + 1$ and if playing $2a + 2b + c = 3k + 2$.

Now, let $V = \{(a, b, c) \mid a, b, c \geq 0, 2a + 2b + c = 3k + 1\}, V' = \{(a, b, c) \mid a, b, c \geq 0, 2a+2b+c = 3k+2\}$. In other words, V is the set of non-playing hands and V' is that of playing hands, and the winning hand in V' is $(1, 0, 0)$. For each turn, if turn started, there are five possibilities: one useless tile, one tile makes a waiting for a sequence with a remaining tile, one tile identical to a remaining tile, waited tile for waiting for a sequence, and an identical tile to a pair. Each can be denoted as adding $(0, 0, 1), (0, 1, -1), (1, 0, -1), (0, -1, 0), (-1, 0, 0)$ from V to V'. Denote this relation as $R \subseteq V \times V'$. Then, we can construct graphs $G = (V, E), G' = (V', E')$ uniquely, which satisfies $G//_R G'$. Moreover, ready hands are nothing but $R^{-1}(1, 0, 0) = \{(2, 0, 0), (1, 1, 0), (0, 0, 1)\}$ and the formula what we want to compute is nothing but $d_G(v, R^{-1}(1, 0, 0))$.

By **Proposition 4**, it is enough to compute $d_{G'}(v, (1, 0, 0))$. Now, in G', possible movements are just combinations of changes by R and R^{-1}, so movements can be shown in the following table.

	$(0,0,1)$	$(0,1,-1)$	$(1,0,-1)$	$(0,-1,0)$	$(-1,0,0)$
$(0,0,-1)$	$(0,0,0)$	$(0,1,-2)$	$(1,0,-2)$	$(0,-1,-1)$	$(-1,0,-1)$
$(0,-1,1)$	$(0,-1,2)$	$(0,0,0)$	$(1,-1,0)$	$(0,-2,1)$	$(-1,-1,1)$
$(-1,0,1)$	$(-1,0,2)$	$(-1,1,0)$	$(0,0,0)$	$(-1,-1,1)$	$(-2,0,1)$
$(0,1,0)$	$(0,1,1)$	$(0,2,-1)$	$(1,1,-1)$	$(0,0,0)$	$(-1,1,0)$
$(1,0,0)$	$(1,0,1)$	$(1,1,-1)$	$(2,0,-1)$	$(1,-1,0)$	$(0,0,0)$

Moreover, if we consider above changes as movements in V', if resulting point is regular then the middle state in V is again regular, so the table above is enough to compute about G'. Here, regular means that every component is a nonnegative integer. For specific example, if $(a, b - 1, c + 2) \in V'$ for some $(a, b, c) \in V'$, then $b \geq 1, a, c \geq 0$. Now, this movement has the middle state $(a, b, c + 1)$, which is automatically in V. This can be checked for every entry in the table above.

Now, we will estimate the distance $d_{G'}$ in G' by the greedy algorithm. First, if $a = 0, c > b - 1$, then repeat $(0, -1, -1)$ b times to reach $(0, 0, c - b)$. Now, $c - b \equiv 2 \pmod 3$. Hence, $c - b \geq 2$, so we can apply (1,0,-2) to reach $(1, 0, c - b - 2)$. Now, repeat $(0, 1, -2)$ and $(0, -1, -1)$ $\frac{c-b-2}{3}$ times, we get $(1, 0, 0)$. Thus, the estimated distance is $b + 1 + \frac{2(c-b-2)}{3} = \frac{b+2c-1}{3}$. For the case when $a \geq 1, c > a+b-1$, we can apply $(0, -1, -1)$ b times and $(-1, 0, -1)$ $(a-1)$ times, and we reach $(1, 0, c - a - b + 1)$. After this, by repeating $(0, 1, -2), (0, -1, -1)$, we get estimated distance $a-1+b+\frac{2(c-a-b+1)}{3}=\frac{a+b+2c-1}{3}$. From these two cases, we estimate the distance as $\frac{a+b+2c-1}{3}$ when $c>a+b-1$.

Now, for the case when $a\geq 1, c\leq a+b-1$, we can apply $(-1,0,-1),(0,-1,-1)$ appropriately, c times in total, we reach $(\alpha,\beta,0)$ with $\alpha\geq 1$. Again, by applying $(-1,-1,0),(-2,0,1),(0,-2,1),(0,-1,-1),(-1,0,-1)$ appropriately, $\frac{2(\alpha+\beta-1)}{3}$ times in total, to reach (1,0,0). Hence, the estimated distance is $\frac{2(\alpha+\beta-1)}{3}+c=\frac{2(a+b-c-1)}{3}+c=\frac{2a+2b+c-2}{3}$. For the case $a=0, c\leq b-1$, we start from $(1,-1,0)$, and this gives the estimated distance $\frac{2a+2b+c+1}{3}$. Thus, the estimated distance is

$$d(a,b,c)=\begin{cases}\frac{a+b+2c-1}{3} & c-a-b\geq 0\\ \\ \frac{2a+2b+c-2}{3}+\max\{0,1-a\} & c-a-b<0.\end{cases}$$

Theorem 5. Above $d(a,b,c)$ is $d_{G'}((a,b,c),(1,0,0))$.

Proof. First, $d(a,b,c)$ is an integer since $2a+2b+c\equiv 2 \pmod 3$. Now, suppose $d(a,b,c)\leq 0$. If $c-a-b\geq 0$, it gives $\frac{a+b+2c-1}{3}\leq 0$, so $0\geq\frac{a+b+2a+2b-1}{3}=a+b-\frac{1}{3}$. Since a,b are nonnegative integers, $0\leq a+b\leq\frac{1}{3}$ gives $a=b=0$. Then, $c\equiv 2\pmod 3$, so $\frac{a+b+2c-1}{3}\geq\frac{4-1}{3}=1>0$, which is a contradiction. Thus, $c-a-b<0$. Then, $\frac{2a+2b+c-2}{3}+\max\{0,1-a\}=0$ and $2a+2b+c\geq 2$ gives $\frac{2a+2b+c-2}{3}=\max\{0,1-a\}=0$. Hence, $a\geq 1, 2a+2b+c=2$. Thus, $a=1, b=c=0$, so $d(a,b,c)\geq 0$ always and $d(a,b,c)=0$ if and only if $(a,b,c)=(1,0,0)$.

Now, for $u,v\in V'$ that are adjacent, compute $|d(u)-d(v)|$. By symmetry, it is enough to check for cases where $u-v$ is one of $(0,1,-2)$, $(1,0,-2)$, $(0,-1,-1)$, $(-1,0,-1)$, $(1,-1,0)$, $(0,-2,1)$, $(-1,-1,1)$, $(-2,0,1)$. First, consider $(0,-1,-1)$, $(-1,0,-1)$, $(1,-1,0)$, where $c-a-b$ is an invariant. The result is in the following table. The table is based on v, and X means we cannot apply given movement in that case.

<table>
<tr><th></th><th>$c-a-b\geq 0$</th><th colspan="3">$c-a-b<0$</th></tr>
<tr><th></th><th></th><th>$a=0$</th><th>$a=1$</th><th>$a\geq 2$</th></tr>
<tr><td>$(0,-1,-1)$</td><td>-1</td><td colspan="3">-1</td></tr>
<tr><td>$(-1,0,-1)$</td><td>-1</td><td>X</td><td>0</td><td>-1</td></tr>
<tr><td>$(1,-1,0)$</td><td>0</td><td>-1</td><td colspan="2">0</td></tr>
</table>

For other cases, we get the following tables.

<table>
<tr><th></th><th>$c-a-b\geq 3$</th><th colspan="2">$c-a-b=2$</th><th colspan="2">$c-a-b=-1$</th><th colspan="2">$c-a-b<-3$</th></tr>
<tr><th></th><th></th><th>$a\geq 1$</th><th>$a=0$</th><th>$a\geq 1$</th><th>$a=0$</th><th>$a\geq 1$</th><th>$a=0$</th></tr>
<tr><td>$(1,0,-2)$</td><td>-1</td><td colspan="2">-1</td><td>0</td><td>-1</td><td>0</td><td>-1</td></tr>
<tr><td>$(0,1,-2)$</td><td>-1</td><td>-1</td><td>0</td><td colspan="4">0</td></tr>
<tr><td>$(0,-2,1)$</td><td colspan="3">0</td><td>0</td><td>-1</td><td colspan="2">-1</td></tr>
</table>

$c-a-b$	≥ 3	$=2$	$=-1$	<-3		
$a \geq 1$				$a \geq 3$	$a=2$	$a=1$
$(-1,-1,1)$	0		0	-1		0
$(-2,0,1)$	0		0	-1	0	X

In any case, $|d(u)-d(v)| \leq 1$.

Lastly, the following case study shows that $d(a,b,c)$ can be reduced in any case except $(1,0,0)$.

- $b,c \geq 1$: Apply $(0,-1,-1)$
- $b=0$:
  - $c-a \geq 0$: Apply $(1,0,-2)$
  - $c-a < 0$:
    - $a \geq 2$:
      - $c=0$: Since $a \geq 4$, so apply $(-2,0,1)$
      - $c>0$: Apply $(-1,0,-1)$
    - $a \leq 1$: Since $c < a \leq 1$, we get $a=1, c=0$. Hence, $(1,0,0)$
- $b \geq 1$, $c=0$:
  - $a=0$: Apply $(1,-1,0)$
  - $a=1$: Since $a+b \geq 4$, $b \geq 3$. Hence, apply $(0,-2,1)$.
  - $a \geq 2$: Apply $(-1,-1,1)$

Thus, by **Proposition 2**, $d(a,b,c) = d_{G'}((a,b,c),(1,0,0))$. $\square$

Thus, the estimated distance formula gives a way to compute the number we want to find in abstracted mahjong.

3 Section 3

Japanese mahjong is one of 13/14 mahjong, which uses all three kinds of suited tiles and honor tiles. In other words, it uses 34 kinds of tiles. For winning hands, there are two exceptional winning hands rather than the usual winning hands, such that 7 pairs and 1, 9 from each suited tiles with every 7 honor tiles and added one more from them. In other words, all of 1 circle, 9 circle, 1 bamboo, 9 bamboo, 1 character, 9 character, east wind, west wind, north wind, south wind, green dragon, red dragon, and white dragon are included in the hand and there exists a pair.

First, we count possible hand types. Since there are 34 kinds of tiles and each tile has 4 copies, by considering the generating function, it is enough to compute the coefficients of $(1+x+x^2+x^3+x^4)^{34}$. The coefficient of x^{13} is 98521596000 and the coefficient of x^{14} is 326520504500. Hence, the number of types of non-playing hands is 98521596000 and of playing hands is 326520504500.

Now, we will count possible winning hand types and the probability of winning at the first turn. If we consider simple estimation, first, we consider the usual winning types. There are 34 kinds of triple, where each triple can be included in the winning hand at most once since each tile has four copies. Also, there are 21 kinds of sequences and each sequence can be included in a winning hand at most four times. Lastly, each winning type has exactly one pair, and there are 34 kinds of pairs. This gives the generating function $(1+x^3)^{34}(1+x^3+x^6+x^9)^{21}(34x^2)$, where this generating function gives an over-estimation since it does not consider the 4-copy-condition perfectly. Hence, we have at most 12663300 types. For seven pairs, there are $\binom{34}{7} = 5379616$ types. Lastly, there are 13 types from another exceptional type.

To compute perfectly, we follow the way of computation based on [1], with the verification of given data in [1]. First, in the winning hand, each type of tile has the number divided by 3 or with the remainder 2. Therefore, we first consider types of winning hands from each type of tile and then, consider the number of combinations of types. The following table gives the number of tiles in a fixed suited type and the number of winning types, which is computed by checking every case by program.

2	9
3	16
5	135
6	127
8	996
9	627
11	4475
12	2098
14	13259

For the honor tiles, the table becomes simpler.

2	7
3	7
5	42
6	21
8	105
9	35
11	140
12	35
14	105

To compute the total number of tiles, we will consider integer solutions of $x_1 + x_2 + x_3 + x_4 = 14$, where each x_i is divided by 3, and exactly one of them has the remainder 2. The following table gives the multisets $\{x_1, x_2, x_3, x_4\}$ and the number of total winning type from a given multiset.

$\{14,0,0,0\}$	$13259*3+105=39882$
$\{12,2,0,0\}$	$2098*9*6+2098*7*3+35*9*3=158295$
$\{11,3,0,0\}$	530295
$\{9,5,0,0\}$	601047
$\{9,3,2,0\}$	1230318
$\{8,6,0,0\}$	861705
$\{8,3,3,0\}$	1514880
$\{6,6,2,0\}$	918210
$\{6,5,3,0\}$	3150234
$\{6,3,3,2\}$	1596000
$\{5,3,3,3\}$	897792

This implies that the total number of winning types is 11498658, which is similar to 12663300. But this also counts 7 pair types, such as 11223355778899, which is also considered as 55/123/123/789/789. For such overlapped cases, by similar computation, we get 4668 types.

Then, to compute the winning probability, we have to consider the weights of each type, which is from that we choose tiles from 4 copies. To count the total number of possible hands with weight, we might consider $(1+4x+6x^2+4x^3+x^4)^{34}=((1+x)^4)^{34}=(1+x)^{136}$, which gives the total number $\binom{136}{14}=$ 4250305029168216000. To compute the total number of types, we will consider the same tables as above. First, for suited tiles, we get

2	54
3	484
5	19200
6	65272
8	1748756
9	2742868
11	47037380
12	40399783
14	440593684

and for honored tiles, we get

2	42
3	28
5	1008
6	336
8	10080
9	2240
11	53760
12	8960
14	161280.

From these tables, the total number of possible types can be computed based on following table

$\{14, 0, 0, 0\}$	1321942332
$\{12, 2, 0, 0\}$	18181353870
$\{11, 3, 0, 0\}$	140625750960
$\{9, 5, 0, 0\}$	324401850432
$\{9, 3, 2, 0\}$	789902278848
$\{8, 6, 0, 0\}$	688605381120
$\{8, 3, 3, 0\}$	1378248505920
$\{6, 6, 2, 0\}$	1234110758400
$\{6, 5, 3, 0\}$	4059699784704
$\{6, 3, 3, 2\}$	2225935514880
$\{5, 3, 3, 3\}$	492095020032,

which gives the total number 11353128141498. For 7 pairs, there are $5379616 - 4668 = 5374948$ pure 7 pairs, and considering weight, we get $5374948 * 6^7 = 1504641443328$ types. For the second exceptional case, we get $13 * 6 * 4^{12} = 1308622848$ types. Hence, the total number of types is 12859078207674, so the probability is $12859078207674/4250305029168216000 = 0.000003025448\cdots$. Moreover, among the winning hands, we can compute the second exceptional case has the probability $1308622848/12859078207674 = 0.000101766\cdots$, which is small, so this gives the reason that the second exceptional cases of hand get high points.

Lastly, we will consider the types of ready hands. For a given ready hand set of tiles, a type is defined by the minimal subset of size $3n+1$ or $3n+2$ such that the waiting structure is preserved. We can consider two kinds of types. One is the *weak type*, which preserves the whole partition structure for waitings. Other is the *strong type*, which preserves only the set of waiting tiles. Note that the weak type of the weak type is itself and the strong type of the strong type is also itself. Hence, we can define the weak type tile set as a tile set with the self-weak type and the strong type tile set as a tile set with self-strong type. Easily, every strong type tile set is a weak type tile set except one case.

In this paper, we will consider types formally. For example, 1111 is treated as a ready hand set where 1 is the waiting tile, and there are two partition structures for waitings, 1/111 and 11/11. It can be easily checked that 1111 is a weak type tile set, and its strong type tile set is 11. I'll mention that this is a very special tile set, since it cannot appear as the weak type tile set of a bigger tile set. All such bigger tile sets have 1111 as the weak type.

Before starting to use the brute-force method, we can reduce the number of cases to check. First, we may assume the tile set only consists of two kinds of suited tiles. Second, by shifting, we may assume the tile set contains the first tiles of suited tile sets that used. Lastly, even the uniqueness of the type of the given ready hand is not proved, a larger tile set that containing given weak or strong type uniquely can be constructed except 11 as weak type. Appendix B contains the detail for the brute-force method.

The number of weak type tile sets with above assumptions is 2090. Only two of them, 1111 and 1111234444, are completely formal, so it is impossible to win, and 11 is the only exception that we cannot find by brute-force method on large tile sets. For strong type tile sets, the number is 745. Since 1111 and 1111234444 are not strong types, every strong type tile is not completely formal where 11 cannot appear for the strong type for an informal ready hand. The full list of types is given in Appendix B. Note that these numbers can be reduced, since our brute-force method counts 1144 and 1155 as distinct type where those are equivalent if we consider partial shifting. Also, if we consider symmetry, we may handle 1112 and 1222 be equivalent to reduce the number.

One interesting kind of strong type tile sets is the one that has every tiles in a suited set as waitings. Using the full list, we can deduce that 2223456677778, 2333344567888, 2344445666678, 1112345666678, 2223456777789, 1233334567888, 2344445678999, and 1112345678999 are only such tile sets. Note that except 1112345678999, every such tile set contains full four copies of some tiles, so 1112345678999 is considered as a very special tile set.

Another interesting kind of weak or strong type tile sets is the tile set that consists of two kinds of suited tiles. Among 2090 weak types, there are 186 types that consist of two kinds of suited tiles. Among 745 strong types, there are 65 such types.

References

[1] http://www10.plala.or.jp/rascalhp/mjmath.htm

Summary

In summary, we have calculated several parameters from mahjong. In Section 2, we proved that the greedy algorithm works with graph theoretical concepts. From this result, we may expect the greedy algorithm to work even when the structure of winning hands is changed. Moreover, if the greedy algorithm always works, then we may construct an algorithm to compute the distance formula from the given structure of winning hands.

Most of the calculations in Section 3 are based on the brute force method rather than the theory. This makes it hard to extend and generalize arguments of Section 3. Hence, from now on, we may try to build up a theory to calculate, or analyze based on some extremal concepts.

Appendix 1

Mahjong is a board game usually played by four people. The goal of mahjong is collecting tiles in some specific ways. Without considering bonus tiles, there are two kinds of tiles: suited tiles and honor tiles. Whether bonus tiles are used or not is determined according to the rules. These two major kinds of tiles are used for almost every mahjong rule, where some of those tiles might not be used based on rules.

Suited tiles are tiles that numbered from one to nine. There are three kinds of suited tiles: circles, bamboo, and characters. For honor tiles, there are two kinds: wind tiles and dragon tiles. Wind tiles denote four cardinal directions, and dragon tiles denote three colors: green, red and white. Lastly, note that for each suited tiles and honor tiles, there are four copies in the whole set.

In the player's hand, three consecutive suited tiles in the same kind is called a sequence. The sequence is not defined for honor tiles. Two identical tiles are called a pair, and three identical tiles are called a triplet. Four identical tiles are not considered as important, but there is a specific rule that players may declare to handle it like a triplet. Lastly, waiting for a sequence means a set of two tiles where one more tile can yield a sequence.

For each turn, the player draws a tile, and if the player cannot win on that turn, the player discards a tile. Therefore, in general, the number of tiles in a player's hand is fixed. Without considering some exceptional winning hand, a general winning hand is defined as a pair, which is called the head, with several triplets and sequences. Most of the rules of mahjong require four of them, so the hand of the playing player of the turn consists of 14 tiles to win, which means non-playing players have 13 tiles in their hand. However, there are some exceptions such as Taiwanese mahjong, which needs five of them.

The non-playing player's hand is called ready hand if the player can satisfy the winning hand condition with only one more tile. Moreover, we consider the *technical ready hand*, which satisfies all conditions of the ready hand but it is impossible to win because of the restriction on the number of tiles. For example, if the hand is 1111777888999, in the same suited kind, it is a technical ready hand since 11/111/777/888/999 is a winning hand, but this is impossible since there is no fifth 1.

Appendix 2

For the brute force methods used in the Section 3, basic idea is that consider a tile set as a sequence of numbers for each tile consisting given tile set rather than as a multiset. For example, tile set 1113456678 from tiles 1 to 9 is considered as the sequence (3,0,1,1,1,2,1,1,0). Then, to process the brute force method, we have to check every such sequence with fixed sum. Hence, we need to consider an algorithm to check every nonnegative integer solutions of $x_1 + \cdots + x_n = m$. Naturally, it is enough to find an algorithm to get the next solution as the output from the given solution as the input. The following algorithm is one of such algorithms.

- The first element is $(0, 0, \cdots, m)$.
- The sequence is $(a_1, \cdots, a_n)$.

$i \leftarrow$ The first nonzero index
$j \leftarrow$ The second nonzero index
if $i = 1$ and $j = \infty$ **then**

    Input sequence is the last.
if $i = 1$ and $j = 2$ **then**
    $(a_1, a_2) \leftarrow (a_1 + 1, a_2 - 1)$
if $i = 1$ and $2 < j < \infty$ **then**
    $(a_1, a_{j-1}, a_j) \leftarrow (0, a_1 + 1, a_j - 1)$
else
    $(a_{i-1}, a_i) \leftarrow (1, a_i - 1)$

This algorithm works since if we consider $b_k = \sum_{i=1}^{k} a_i$, then the algorithm gives every nonnegative nondecreasing sequence $b_1, \cdots, b_{m-1}$ satisfying $b_{m-1} \leq m$ in colexicographic order. If we introduce a temporary variable, this algorithm can be reduced as following.

$i \leftarrow$ The first nonzero index satisfying $i \geq 2$.
if $i = \infty$ **then**
    Input sequence is the last.
else
    temp$\leftarrow a_1$
    $a_1 \leftarrow 0$
    $a_{i-1} \leftarrow$temp+1
    $a_i \leftarrow a_i - 1$

Equivalence of two algorithms can be checked easily.

Now, the following list is the full list of 745 strong type tile sets that consist of at most two types of tiles and contain the first tile of consisting types.

1
11
1111222233
1111222233345
1111222233456
1111222333
1111222333344
1111222333345
1111222333456
1111223333445
1111223333455
1111223334
1111223334456
1111223334555
1111223334567
1111223344566
1111233
1111233334455
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1111233345555
1111233345678
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1111233456
1111233456678
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1111233555
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1112
1112222333444
1112222333445
1112222333455
1112222334
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1112222334567
1112223
1112223333
1112223333444
1112223333445
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1112223334
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1112223334455
1112223334456
1112223334555
1112223334567
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1112223334588
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1112223355567
1112223356777
1112223366
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1112223377
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1112223378999
1112223388
1112223399
11122233aa
11122233aaabc
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11123abccc
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1233456888
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1233567888
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1234444566789

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1234445556667
1234445556677
1234445556688
1234445556699
12344455566aa
1234445566
1234445566667
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1234566699
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1234567788999
1234567789
1234567789999
12345678
1234567888
1234567888899
1234567888999
12345678999aa
1234567999
1234666
1234677788899
1234678999
13
1333
1344455566
1345666
1345677788899
1345678999

The following list is the full list of 2090 weak type tile sets.

1
11
1111
1111222
1111222233
1111222233334
1111222233344
1111222233345
1111222233444
1111222233445
1111222233456
1111222233555
1111222234
1111222234444
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