Four identities

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1 Result

We have following identities for any $k, n \geq 1$ where the empty product is 1.

$$x^k = \sum_{t=n}^{\infty}(1 - x^k - x^t)\prod_{s=n}^{t}\frac{x^k}{1-x^s} \tag{1}$$

$$x^k \sum_{j=k}^{\infty}\left(\prod_{i=k}^{j}\frac{x^n}{1-x^i}\right) = x^n \sum_{t=n}^{\infty}\left(\prod_{s=n}^{t}\frac{x^k}{1-x^s}\right) \tag{2}$$

$$1 + \sum_{t=1}^{\infty}\prod_{s=1}^{t}\frac{x^k}{1-x^s} = \sum_{t=0}^{\infty}\prod_{s=1}^{t}\frac{x^k}{1-x^s} = \prod_{m=k}^{\infty}\frac{1}{1-x^m} \tag{3}$$

$$\left(\prod_{i=k}^{\infty}\frac{1}{1-x^i}\right)\left(\sum_{j=k}^{\infty}\frac{x^j}{1-x^j}\right) = \sum_{t=1}^{\infty} t \prod_{s=1}^{t}\frac{x^k}{1-x^s} \tag{4}$$

In q-Pochhammer symbols, it is equivalent to

$$q^k = \sum_{t=n}^{\infty}(1-q^k-q^t)\frac{q^{kt}}{(q^n;q)_{t-n+1}} = \sum_{s=1}^{\infty}(1-q^k-q^{s+n-1})\frac{q^{k(s+n-1)}}{(q^n;q)_s}$$

$$q^{kn}\sum_{i=1}^{\infty}\frac{q^{n(i-1)}}{(q^k;q)_i} = \sum_{j=k}^{\infty}\frac{q^{nj}}{(q^k;q)_{j-k+1}} = \sum_{t=n}^{\infty}\frac{q^{kt}}{(q^n;q)_{t-n+1}} = q^{kn}\sum_{s=1}^{\infty}\frac{q^{k(s-1)}}{(q^n;q)_s}$$

$$1 + \sum_{t=1}^{\infty}\frac{q^{kt}}{(q;q)_t} = \sum_{t=0}^{\infty}\frac{q^{kt}}{(q;q)_t} = \frac{1}{(q^k;q)_\infty}$$

$$\frac{1}{(q^k;q)_\infty}\left(\sum_{j=k}^{\infty}\frac{q^j}{1-q^j}\right) = \sum_{t=1}^{\infty}\frac{tq^{kt}}{(q;q)_t}$$

2 Combinatorial approach

At first, let Λ be the set of all integer partition λ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0 \geq \cdots$, λ' be the conjugate of λ which is $\lambda'_i = \#\{j \mid \lambda_j \geq i\}$, and $s(\lambda) = \sum_i \lambda_i < \infty$ so $\lambda \vdash s(\lambda)$. For any integer partition λ, define $\lambda^{(d,r)}$ as

$$\lambda_m^{(d,r)} = \begin{cases} \lambda_m + r & \text{if } m \leq d \\ \lambda_m & \text{otherwise} \end{cases}$$

and

$$\lambda_m^{(i:j)} = \begin{cases} \lambda_{i+m-1} & \text{if } m \leq j-i+1 \\ 0 & \text{otherwise} \end{cases}$$

so we have $\lambda^{(i:j)}$ is $\lambda_i \geq \cdots \geq \lambda_j \geq 0 \geq \cdots$. Then, define

$$\begin{aligned} \Lambda_{k:t} &= \{\lambda \in \Lambda \mid \lambda_k = t\} \\ \Lambda_{k:\leq t} &= \{\lambda \in \Lambda \mid \lambda_k \leq t\} \\ \Lambda_{k:\geq t} &= \{\lambda \in \Lambda \mid \lambda_k \geq t\} \end{aligned}$$

Then, for any $A \subseteq \Lambda$, we may define the generating function $P(A)$ as

$$P(A)(x) = \sum_{\lambda \in A} x^{s(\lambda)} = \sum_{n=0}^{\infty} \#\{\lambda \in A \mid s(\lambda) = n\} x^n$$

naturally. The following is well-known.

Lemma 1. For any $I \subseteq \mathbb{N}$, if we define $\Lambda_I = \{\lambda \in \Lambda \mid \lambda_i \in I \text{ if } \lambda_i \neq 0\}$, then

$$P(\Lambda_I) = \prod_{i \in I} \frac{1}{1-x^i}$$

Note that $\Lambda_{1:\leq s} = \Lambda_{\{1,2,\cdots,s\}}$ since integer partitions are monotone decreasing sequences, so we have

Corollary 2.

(i) For any $s \geq 1$,

$$P(\Lambda_{1:\leq s})(x) = \prod_{i=1}^{s} \frac{1}{1-x^i} =: P_s(x)$$

(ii) For Λ, we have

$$P(\Lambda)(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i} =: P_\infty(x)$$

where we define $P_0(x) = 1$.

Also, we have simple well-known property of the conjugation.

Lemma 3. The conjugation is a s-preserving bijection from $\Lambda_{k:\geq n}$ to $\Lambda_{n:\geq k}$ for every $k, n \geq 1$.

Proof. Note that $\lambda'_i = \#\{j \mid \lambda_j \geq i\} = \sum_{j=1}^{\infty} 1_{\lambda_j \geq i}$. Then, we have

$$\sum_{i=1}^{\infty} \lambda'_i = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} 1_{\lambda_j \geq i} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} 1_{\lambda_j \geq i} = \sum_{j=1}^{\infty} \lambda_j$$

since λ_j's are nonnegative integers, which proves $s(\lambda) = s(\lambda')$. Hence, conjugating is a s-preserving operation.

Now, since λ is a decreaing sequence, so $\lambda_k \geq n$ implies $\lambda_1 \geq \cdots \geq \lambda_k \geq n$, so $\{1, \cdots, k\} \subseteq \{j \mid \lambda_j \geq n\}$ which proves $\lambda'_n \geq k$. Hence, the conjugation is a map from $\Lambda_{k:\geq n}$ to $\Lambda_{n:\geq k}$. Also, if $\lambda_k < n$, then it implies $\{j \mid \lambda_j \geq n\} \subseteq \{1, 2, \cdots, k-1\}$ so $\lambda'_n < k$. Thus, $\lambda'_n \geq k$ is actually equivalent to $\lambda_k \geq n$. Then, it gives

$$\lambda''_j = \#\{i \mid \lambda'_i \geq j\} = \#\{i \mid \lambda_j \geq i\} = \lambda_j$$

which proves $\lambda'' = \lambda$, so the conjugation is a bijection. □

Then, it will conclude $\lambda_k \leq n-1$ is equivalent to $\lambda_n \leq k-1$ also. Now, we can find out the generating function of $\Lambda_{k:t}$.

Theorem 4. For $k \geq 1$ and $t \geq 0$, we have

$$P(\Lambda_{k:t})(x) = x^{kt} P_{k-1}(x) P_t(x)$$

Proof. We will construct the bijection $\phi : \Lambda_{k:t} \to \Lambda_{1:\leq k-1} \times \Lambda_{1:\leq t}$ such that if $\phi(\lambda) = (\mu, \nu)$, then $s(\lambda) = s(\mu) + s(\nu) + kt$. From this, we will get

$$\begin{aligned}
\sum_{\lambda \in \Lambda^{k:t}} x^{s(\lambda)} &= \sum_{\mu \in \Lambda_{1:\leq k-1}, \nu \in \Lambda_{1:\leq t}} x^{s(\phi^{-1}(\mu,\nu))} \\
&= \sum_{\mu \in \Lambda_{1:\leq k-1}} \sum_{\nu \in \Lambda_{1:\leq t}} x^{s(\mu)+s(\nu)+kt} \\
&= x^{kt} \left(\sum_{\mu \in \Lambda_{1:\leq k-1}} x^{s(\mu)} \right) \left(\sum_{\nu \in \Lambda_{1:\leq t}} x^{s(\nu)} \right) \\
&= x^{kt} P_{k-1}(x) P_t(x)
\end{aligned}$$

which completes the proof.

First, if $\lambda_k = t$, then we have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k-1} \geq \lambda_k = t$, so

$$\mu = (\lambda^{(1:k-1)})^{(k-1,-t)} = \lambda_1 - t \geq \cdots \geq \lambda_{k-1} - t \geq 0 \cdots$$

is well-defined integer partition. Note that if $k = 1$, then μ is the zero partition. Moreover, $\mu_k = 0$ so we have $\mu'_1 \leq k-1$. Thus, $\mu' \in \Lambda_{1:\leq k-1}$. Now, $t = \lambda_k \geq \lambda_{k+1} \geq \cdots$, so $\nu = \lambda^{(k+1:\infty)} \in \Lambda_{1:\leq t}$. Hence, the map

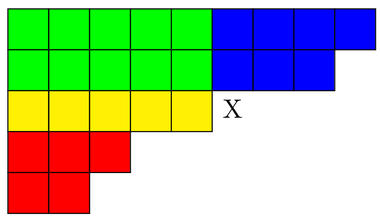
$$\phi(\lambda) = (\mu', \nu) = ((\lambda^{(1:k-1)})^{(k-1,-t)\prime}, \lambda^{(k+1:\infty)})$$

is well-defined map from $\Lambda_{k:t}$ to $\Lambda_{1:\le k-1} \times \Lambda_{1:\le t}$. Naturally, $s(\mu') = s(\mu) = s(\lambda^{(1:k-1)}) - t(k-1)$ and $s(\nu) = s(\lambda^{(k+1:\infty)})$. Thus, we have

$$\begin{aligned} s(\lambda) &= s(\lambda^{(1:k-1)}) + \lambda_k + s(\lambda^{(k+1:\infty)}) \\ &= s(\mu') + t(k-1) + t + s(\nu) \\ &= s(\mu') + s(\nu) + kt \end{aligned}$$

whenever $\phi(\lambda) = (\mu', \nu)$. Hence, it is enough to prove that ϕ is bijective. Injectivity is natural since $\lambda^{(1:k-1)} = \kappa^{(1:k-1)}$ and $\lambda^{(k+1:\infty)} = \kappa^{(k+1:\infty)}$ implies $\lambda = \kappa$ since $\lambda_k = \kappa_k = t$. For surjectivity, you may note that $\phi(\mu'_1 + t \geq \mu'_2 + t \geq \cdots \geq \mu'_{k-1} + t \geq t \geq \nu_1 \geq \nu_2 \geq \cdots) = (\mu, \nu)$ for any given $\mu \in \Lambda_{1:\le k-1}$ and $\nu \in \Lambda_{1:\le t}$.

We can understand this proof visually.



The yellow area denotes $\lambda_k = t$. X denotes the position which will never be the cell of the young diagram of integer partition λ with $\lambda_k = t$. The blue area is corresponding to μ, where X makes $\mu'_1 \leq k-1$ so $\mu' \in \Lambda_{1:\le k-1}$, so it corresponding to P_{k-1} for the generating function. The red area is corresponding to $\nu \in \Lambda_{1:\le t}$, so it corresponding to P_t for the generating function. Lastly, the number of green and yellow cell is exactly kt, which makes x^{kt} in the generating function. □

Now, we can prove equation (1), (2) and (3). For convinience, define

$$P_{k,n} = \prod_{i=1}^{n} \frac{x^k}{1-x^i} = x^{kn} P_n$$

Theorem 5. The following identities are true.

(i)

$$x^k = \sum_{t=n}^{\infty} (1 - x^k - x^t) \prod_{s=n}^{t} \frac{x^k}{1-x^s}$$

(ii)

$$x^k \sum_{j=k}^{\infty} \left(\prod_{i=k}^{j} \frac{x^n}{1-x^i} \right) = x^n \sum_{t=n}^{\infty} \left(\prod_{s=n}^{t} \frac{x^k}{1-x^s} \right)$$

(iii)

$$1 + \sum_{t=1}^{\infty} \prod_{s=1}^{t} \frac{x^k}{1-x^s} = \sum_{t=0}^{\infty} \prod_{s=1}^{t} \frac{x^k}{1-x^s} = \prod_{m=k}^{\infty} \frac{1}{1-x^m}$$

Proof. (i) First, $\Lambda_{n:k} = \Lambda_{n:\geq k} \setminus \Lambda_{n:\geq k+1}$. Hence, by **Lemma 3.**, we have

$$P(\Lambda_{n:k}) = P(\Lambda_{n:\geq k}) - P(\Lambda_{n:\geq k+1}) = P(\Lambda_{k:\geq n}) - P(\Lambda_{k+1:\geq n})$$

which gives

$$\begin{aligned} x^{kn} P_{n-1} P_k &= \sum_{t=n}^{\infty} x^{kt} P_{k-1} P_t - \sum_{t=n}^{\infty} x^{(k+1)t} P_k P_t \\ &= \sum_{t=n}^{\infty} x^{kt}(1-x^k) P_k P_t - \sum_{t=n}^{\infty} x^{(k+1)t} P_k P_t \\ &= \sum_{t=n}^{\infty} x^{kt}(1-x^k-x^t) P_k P_t. \end{aligned}$$

Hence,

$$x^k P_{k,n-1} = x^{kn} P_{n-1} = \sum_{t=n}^{\infty} x^{kt}(1-x^k-x^t) P_t = \sum_{t=n}^{\infty} (1-x^k-x^t) P_{k,t}$$

Thus, we have

$$\begin{aligned} x^k &= \frac{1}{P_{k,n-1}} \sum_{t=n}^{\infty} (1-x^k-x^t) P_{k,t} \\ &= \sum_{t=n}^{\infty} (1-x^k-x^t) \frac{P_{k,t}}{P_{k,n-1}} \\ &= \sum_{t=n}^{\infty} (1-x^k-x^t) \prod_{s=n}^{t} \frac{x^k}{1-x^s} \end{aligned}$$

□

(ii) From **Lemma 3.**, we have $P(\Lambda_{n:\geq k}) = P(\Lambda_{k:\geq n})$. Hence,

$$\sum_{j=k}^{\infty} x^{jn} P_{n-1} P_j = \sum_{t=n}^{\infty} x^{kt} P_{k-1} P_t$$

Divide both side by $x^{kn-k-n} P_{n-1} P_{k-1}$, then we get

$$\begin{aligned} x^k \sum_{j=k}^{\infty} \frac{P_{n,j}}{P_{n,k-1}} &= x^k \sum_{j=k}^{\infty} \frac{x^{jn} P_j}{x^{(k-1)n} P_{k-1}} \\ &= x^n \sum_{t=n}^{\infty} \frac{x^{kt} P_t}{x^{k(n-1)} P_{n-1}} \\ &= x^n \sum_{t=n}^{\infty} \frac{P_{k,t}}{P_{k,n-1}}. \end{aligned}$$

Thus,

$$x^k \sum_{j=k}^{\infty} \left(\prod_{i=k}^{j} \frac{x^n}{1-x^i} \right) = x^n \sum_{t=n}^{\infty} \left(\prod_{s=n}^{t} \frac{x^k}{1-x^s} \right)$$

□

(iii) Since $\Lambda = \dot\bigcup_{t=0}^{\infty} \Lambda_{k:t}$, we have

$$\sum_{t=0}^{\infty} x^{kt} P_{k-1} P_t = P_\infty$$

Hence,

$$\sum_{t=0}^{\infty} \prod_{s=1}^{t} \frac{x^k}{1-x^s} = \sum_{t=0}^{\infty} P_{k,t} = \frac{P_\infty}{P_{k-1}} = \prod_{m=k}^{\infty} \frac{1}{1-x^m} \quad \square$$

To prove (4), we need one more bijection, which is from [1].

Lemma 6. Suppose $d_k(m) = \{d \mid d \geq k, d \mid m\}$ and $p(n) = [x^n]P_\infty(x) = \#\{\lambda \in \Lambda \mid \lambda \vdash n\}$. Then,

$$\sum_{\lambda \vdash n} \lambda_k = \sum_{m=1}^{\infty} p(n-m) d_k(m)$$

Proof. For any triple (λ, d, r) such that λ is an integer paritition with $\lambda \vdash n$, $d \geq k$, $r \geq 1$, and $\lambda^{(d,-r)}$ is again an integer partition, define $\psi(\lambda, d, r) = (dr, \lambda^{(d,-r)}, d)$, which is a bijective to the set of triples (m, κ, d) such that $m \geq 1$, κ is an integer partition with $\kappa \vdash n - m$, $d \geq k$, and d divides m since $(m, \kappa, d) \mapsto (\kappa^{(d,m/d)}, d, m/d)$ is the inverse map.

For fixed $\lambda \vdash n$ and $d \geq k$, the number of triple (λ, d, r) is $\lambda_d - \lambda_{d+1}$ since $\lambda^{(d,-r)}$ is an integer paritition if and only if $\lambda_d - r \geq \lambda_{d+1}$. Hence, the number of triples (λ, d, r) for fixed λ is $\sum_{d=k}^{\infty}(\lambda_d - \lambda_{d+1}) = \lambda_k$ and thus, the number of triples (λ, d, r) is $\sum_{\lambda \vdash n} \lambda_k$. Now, for fixed $m \geq 1$, the number of triples (m, κ, d) is $p(n-m)d_k(m)$, so we get

$$\sum_{\lambda \vdash n} \lambda_k = \sum_{m=1}^{\infty} p(n-m) d_k(m)$$

□

Also, we can get the generating function of d_k, which is defined as $D_k(x) = \sum_{n=1}^{\infty} d_k(n) x^n$, easily.

Lemma 7. For any $k \geq 1$,

$$D_k(x) = \sum_{d=k}^{\infty} \frac{x^d}{1-x^d}$$

Proof. Easily,

$$\begin{aligned}\sum_{n=1}^{\infty} d_k(n)x^n &= \sum_{n=1}^{\infty} \#\{d \mid d \geq k, d \mid n\}x^n \\ &= \sum_{n=1}^{\infty}\sum_{d=k}^{\infty} 1_{d|n}x^n \\ &= \sum_{d=k}^{\infty}\sum_{n=1}^{\infty} 1_{d|n}x^n \\ &= \sum_{d=k}^{\infty}\sum_{i=1}^{\infty} x^{di} \\ &= \sum_{d=k}^{\infty} \frac{x^d}{1-x^d}\end{aligned}$$

□

Then, we are ready to prove (4)

Theorem 8. The following identity is true

$$\left(\prod_{i=k}^{\infty} \frac{1}{1-x^i}\right)\left(\sum_{j=k}^{\infty} \frac{x^j}{1-x^j}\right) = \sum_{t=1}^{\infty} t \prod_{s=1}^{t} \frac{x^k}{1-x^s}$$

Proof. From above lemmas, we get

$$\sum_{n=1}^{\infty}\sum_{\lambda\vdash n} \lambda_k x^n = D_k(x)P_\infty(x)$$

Note that $\sum_{\lambda\vdash n}\lambda_k = \sum_{t=1}^{\infty} t\#\{\lambda \vdash n \mid \lambda_k = t\}$, so

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{\lambda\vdash n} \lambda_k x^n &= \sum_{n=1}^{\infty}\sum_{t=1}^{\infty} t\#\{\lambda \vdash n \mid \lambda_k = t\}x^n \\ &= \sum_{t=1}^{\infty} t \sum_{n=1}^{\infty} \#\{\lambda \vdash n \mid \lambda_k = t\}x^n \\ &= \sum_{t=1}^{\infty} tP(\Lambda_{k:t})(x) \\ &= \sum_{t=1}^{\infty} tx^{kt}P_{k-1}(x)P_t(x)\end{aligned}$$

Thus,

$$\frac{P_\infty}{P_{k-1}}D_k(x) = \sum_{t=1}^{\infty} tP_{k,t}$$

which is given identity. □

3 Algebraic approach

In this section, we will prove given identity algebraically.

Theorem 9. The following identities are true.

(i)

$$x^k = \sum_{t=n}^{\infty}(1 - x^k - x^t)\prod_{s=n}^{t}\frac{x^k}{1-x^s}$$

(ii)

$$x^k \sum_{m=k}^{\infty}\left(\prod_{i=k}^{m}\frac{x^n}{1-x^i}\right) = x^n \sum_{t=n}^{\infty}\left(\prod_{s=n}^{t}\frac{x^k}{1-x^s}\right)$$

Proof. (i) By computation,

$$\begin{aligned}\sum_{t=n}^{\infty}(1 - x^k - x^t)\prod_{s=n}^{t}\frac{x^k}{1-x^s} &= \sum_{t=n}^{\infty}(1 - x^t)\prod_{s=n}^{t}\frac{x^k}{1-x^s} - x^k\sum_{t=n}^{\infty}\prod_{s=n}^{t}\frac{x^k}{1-x^s}\\ &= (1-x^t)\frac{x^k}{1-x^t} + \sum_{t=n+1}^{\infty}x^k\prod_{s=n}^{t-1}\frac{x^k}{1-x^s} - x^k\sum_{t=n}^{\infty}\prod_{s=n}^{t}\frac{x^k}{1-x^s}\\ &= x^k\end{aligned}$$

□

(ii) By (i)

$$x^j = \sum_{t=n}^{\infty}(1 - x^j - x^t)\prod_{s=n}^{t}\frac{x^j}{1-x^s}$$

Then, for $j \geq k$, we have

$$\begin{aligned}x^k\prod_{i=k}^{j}\frac{x^n}{1-x^i} &= \left(\frac{x^k}{x^j}\prod_{i=k}^{j}\frac{x^n}{1-x^i}\right)\sum_{t=n}^{\infty}(1-x^j-x^t)\prod_{s=n}^{t}\frac{x^j}{1-x^s}\\ &=\frac{x^k}{x^j}\sum_{t=n}^{\infty}(1-x^j-x^t)\prod_{i=k}^{j}\frac{x^n}{1-x^i}\prod_{s=n}^{t}\frac{x^j}{1-x^s}\\ &=\frac{x^k}{x^j}\sum_{t=n}^{\infty}(1-x^j)\prod_{i=k}^{j}\frac{x^n}{1-x^i}\prod_{s=n}^{t}\frac{x^j}{1-x^s}\\ &\quad-\frac{x^k}{x^j}\sum_{t=n}^{\infty}x^t\prod_{i=k}^{j}\frac{x^n}{1-x^i}\prod_{s=n}^{t}\frac{x^j}{1-x^s}\end{aligned}$$

$$= \sum_{t=n}^{\infty} x^n \prod_{i=k}^{j-1} \frac{x^{n-1}}{1-x^i} \prod_{s=n}^{t} \frac{x^j}{1-x^s}$$

$$- \sum_{t=n}^{\infty} x^n \prod_{i=k}^{j} \frac{x^{n-1}}{1-x^i} \prod_{s=n}^{t} \frac{x^{j+1}}{1-x^s}$$

Thus, we get

$$x^k \sum_{j=k}^{\infty} \prod_{i=k}^{j} \frac{x^n}{1-x^i} = x^n \sum_{t=n}^{\infty} \prod_{i=k}^{k-1} \frac{x^{n-1}}{1-x^i} \prod_{s=n}^{t} \frac{x^k}{1-x^s} = x^n \sum_{t=n}^{\infty} \prod_{s=n}^{t} \frac{x^k}{1-x^s}$$

□

To prove (3), we have to define some special polynomials.

Definition 10. For any $k \geq 1, m \geq 0$, define

$$Q_{k+1,m}(x) = Q_{k,m}(x)(1-x^k) + x^{(k-1)(m+1)}$$

with

$$Q_{1,m}(x) = 0$$

and

$$N_{k,m}(x) = \prod_{i=1}^{k-1}(1-x^i) + x^{m+1} Q_{k,m}(x)$$

Lemma 11. Polynomials Q and N have following properties.

(i)

$$N_{k+1,m}(x) = N_{k,m}(x)(1-x^k) + x^{k(m+1)}$$

(ii)

$$N_{k,m+1}(x) = N_{k,m}(x)(1-x^{m+1}) + x^{k(m+1)}$$

(iii)

$$N_{k,0}(x) = 1$$

(iv)

$$\sum_{t=0}^{m} \prod_{s=1}^{t} \frac{x^k}{1-x^s} = \frac{N_{k,m}(x)}{\prod_{s=1}^{m}(1-x^s)}$$

Proof. (i) By definition,

$$N_{k+1,m}(x) = \prod_{i=1}^{k}(1-x^i) + x^{m+1} Q_{k+1,m}(x)$$

$$= (1-x^k) \prod_{i=1}^{k-1}(1-x^i) + x^{m+1} Q_{k,m}(x)(1-x^k) + x^{k(m+1)}$$

$$= (1-x^k) \left(\prod_{i=1}^{k-1}(1-x^i) + x^{m+1} Q_{k,m}(x) \right) + x^{k(m+1)}$$

$$= (1 - x^k) N_{k,m}(x) + x^{k(m+1)}$$

$\square$

(ii) We will use mathematical induction. First, from $Q_{1,m}(x) = 0$, we have $N_{1,m}(x) = 1$. Hence, it is true when $k = 1$. Now, assume it is true for k. Then,

$$\begin{aligned}
N_{k+1,m+1}(x) &= N_{k,m+1}(x)(1 - x^k) + x^{k(m+2)} \\
&= N_{k,m}(x)(1 - x^{m+1})(1 - x^k) + x^{k(m+1)}(1 - x^k) + x^{k(m+2)} \\
&= N_{k,m}(x)(1 - x^k)(1 - x^{m+1}) + x^{k(m+1)} \\
&= \left(N_{k,m}(x)(1 - x^k) + x^{k(m+1)}\right)(1 - x^{m+1}) + x^{(k+1)(m+1)} \\
&= N_{k+1,m}(x)(1 - x^{m+1}) + x^{(k+1)(m+1)}
\end{aligned}$$

by (i). Thus, done. $\square$

(iii) We have $N_{1,0}(x) = 1$. Now, if we have $N_{k,0}(x) = 1$, then by (i), we have $N_{k+1,0}(x) = 1$, so is done by mathematical induction. $\square$

(iv) From (iii), it is true when $m = 0$. Now, if it is true for m, then

$$\begin{aligned}
\sum_{t=0}^{m+1} \prod_{s=1}^{t} \frac{x^k}{1 - x^s} &= \frac{N_{k,m}(x)}{\prod_{s=1}^{m}(1 - x^s)} + \frac{x^{k(m+1)}}{\prod_{s=1}^{m+1}(1 - x^s)} \\
&= \frac{N_{k,m}(x)(1 - x^{m+1}) + x^{k(m+1)}}{\prod_{s=1}^{m+1}(1 - x^s)} \\
&= \frac{N_{k,m+1}(x)}{\prod_{s=1}^{m+1}(1 - x^s)}
\end{aligned}$$

Thus, done by mathematical induction. $\square$

Then, it gives (3) and (4)

Theorem 12. The following identities are true.

(i)

$$1 + \sum_{t=1}^{\infty} \prod_{s=1}^{t} \frac{x^k}{1 - x^s} = \sum_{t=0}^{\infty} \prod_{s=1}^{t} \frac{x^k}{1 - x^s} = \prod_{m=k}^{\infty} \frac{1}{1 - x^m}$$

(ii)

$$\left(\prod_{i=k}^{\infty} \frac{1}{1 - x^i}\right) \left(\sum_{j=k}^{\infty} \frac{x^j}{1 - x^j}\right) = \sum_{t=1}^{\infty} t \prod_{s=1}^{t} \frac{x^k}{1 - x^s}$$

Proof. (i) By previous lemma, we have $\sum_{t=0}^{m}\prod_{s=1}^{t}\frac{x^k}{1-x^s}=\frac{N_{k,m}}{\prod_{s=1}^{m}(1-x^s)}$. Since $N(k,m)(x)=\prod_{i=1}^{k-1}(1-x^i)+x^{m+1}Q_{k,m}(x)$, when we consider limit, we have $\lim_{m\to\infty}N(k,m)(x)=\prod_{i=1}^{k-1}(1-x^i)$. Thus, we have

$$\sum_{t=0}^{\infty}\prod_{s=1}^{t}\frac{x^k}{1-x^s}=\frac{\prod_{i=1}^{k-1}(1-x^i)}{\prod_{s=1}^{\infty}(1-x^s)}=\prod_{m=k}^{\infty}\frac{1}{1-x^m}$$

□

(ii) Now, by (2) and (i),

$$\begin{aligned}
\sum_{t=1}^{\infty}t\prod_{s=1}^{t}\frac{x^k}{1-x^s} &= \sum_{n=1}^{\infty}\sum_{t=n}^{\infty}\prod_{s=1}^{t}\frac{x^k}{1-x^s}\\
&= \sum_{n=1}^{\infty}\prod_{s=1}^{n-1}\frac{x^k}{1-x^s}\left(\sum_{t=n}^{\infty}\prod_{s=n}^{t}\frac{x^k}{1-x^s}\right)\\
&= \sum_{n=1}^{\infty}\frac{x^k}{x^n}\prod_{s=1}^{n-1}\frac{x^k}{1-x^s}\left(\sum_{j=k}^{\infty}\prod_{i=k}^{j}\frac{x^n}{1-x^i}\right)\\
&= \sum_{n=1}^{\infty}\sum_{j=k}^{\infty}\frac{x^k}{x^n}\prod_{s=1}^{n-1}\frac{x^k}{1-x^s}\prod_{i=k}^{j}\frac{x^n}{1-x^i}\\
&= \sum_{j=k}^{\infty}\sum_{n=1}^{\infty}x^j\prod_{i=k}^{j}\frac{1}{1-x^i}\prod_{s=1}^{n-1}\frac{x^j}{1-x^s}\\
&= \sum_{j=k}^{\infty}x^j\prod_{i=k}^{j}\frac{1}{1-x^i}\left(\sum_{n=1}^{\infty}\prod_{s=1}^{n-1}\frac{x^j}{1-x^s}\right)\\
&= \sum_{j=k}^{\infty}x^j\prod_{i=k}^{j}\frac{1}{1-x^i}\left(\sum_{n=0}^{\infty}\prod_{s=1}^{n}\frac{x^j}{1-x^s}\right)\\
&= \sum_{j=k}^{\infty}x^j\prod_{i=k}^{j}\frac{1}{1-x^i}\prod_{m=j}^{\infty}\frac{1}{1-x^m}\\
&= \sum_{j=k}^{\infty}\frac{x^j}{1-x^j}\prod_{i=k}^{\infty}\frac{1}{1-x^i}\\
&= \left(\prod_{i=k}^{\infty}\frac{1}{1-x^i}\right)\left(\sum_{j=k}^{\infty}\frac{x^j}{1-x^j}\right)
\end{aligned}$$

References

[1] Kimmo Eriksson, *A note on the exact expected length of the kth part of a random partition*, Integers, Vol.10, 309-311 (2010).