2-element dependency on four pemutations

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derived from joint work with
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Permutation Puzzle

4 permutations

7	4	6	1	5	2	3
6	5	4	2	3	7	1
2	5	4	6	3	1	7
1	2	3	4	5	6	7

Permutation Puzzle

Select 3,4

7	4	6	1	5	2	3
6	5	4	2	3	7	1
2	5	4	6	3	1	7
1	2	3	4	5	6	7

Permutation Puzzle

Color left

7	4	6	1	5	2	3
6	5	4	2	3	7	1
2	5	4	6	3	1	7
1	2	3	4	5	6	7

Permutation Puzzle

Common elements

7	4	6	1	5	2	3
6	5	4	2	3	7	1
2	5	4	6	3	1	7
1	2	3	4	5	6	7

Permutation Puzzle

Select 4,5

7	4	6	1	5	2	3
6	5	4	2	3	7	1
2	5	4	6	3	1	7
1	2	3	4	5	6	7

Permutation Puzzle

Color left

7	4	6	1	5	2	3
6	5	4	2	3	7	1
2	5	4	6	3	1	7
1	2	3	4	5	6	7

Permutation Puzzle

Common elements

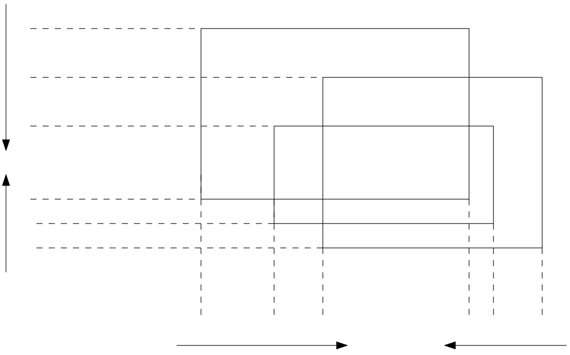
7	4	6	1	5	2	3
6	5	4	2	3	7	1
2	5	4	6	3	1	7
1	2	3	4	5	6	7

Permutation Puzzle

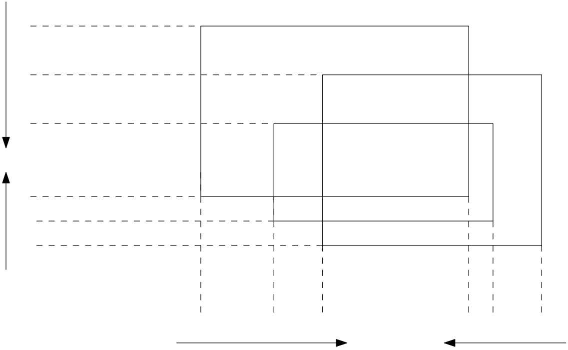
No blue elements for any choice

6	8	5	7	2	4	1	3
7	5	8	6	3	1	4	2
4	3	2	1	8	7	6	5
1	2	3	4	5	6	7	8

Geometric Motivation

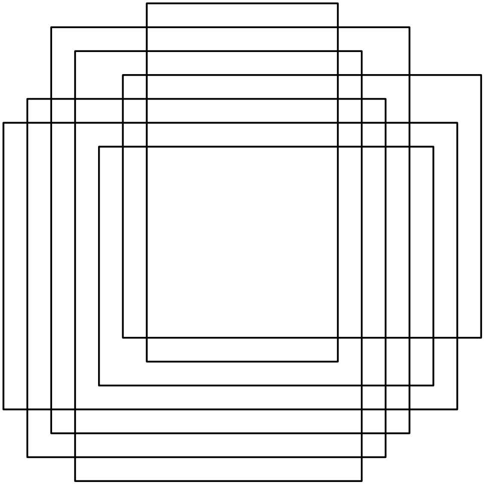


Geometric Motivation

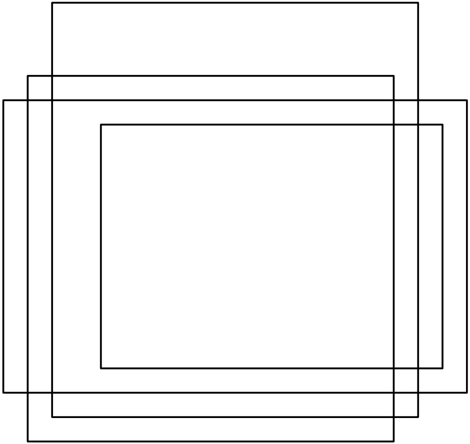


Blue elements indicate boxes which cover the intersection of selected boxes.

Example Revisited



Example Revisited



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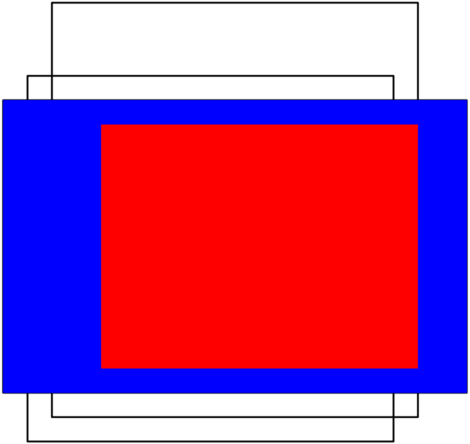

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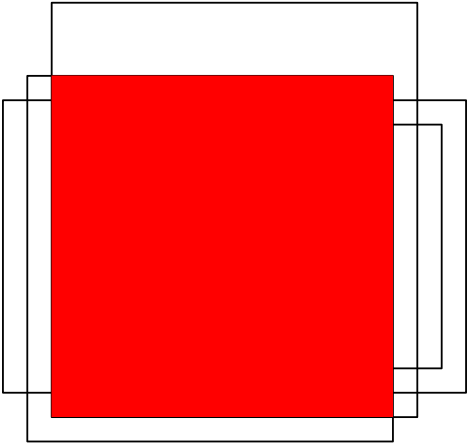

Question

For given $a \geq 3$ and $p \geq 2$, if n is sufficiently large, then any a permutations in S_n have a proper choice of p red elements such that there exists at least one blue element?

Answer (E., Kim and Lee)

There exists $n(a, b; p)$ such that if $n \geq n(a, b; p)$, then any a permutations in S_n have a proper choice of p red elements to get at least b blue elements.

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$n(a, b; p) \leq n(a, b + p - 2; 2)$, so we only need to check when $p = 2$.

Pattern Graph

12

σ_0 :	7	4	6	1	5	2	3
σ_1 :	6	5	4	2	3	7	1
σ_2 :	2	5	4	6	3	1	7
id :	1	2	3	4	5	6	7

Pattern Graph

12

σ_0 :	7	4	6	1	5	2	3
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$12 \to 2^0 = 1$

Pattern Graph

23

σ_0 :	7	4	6	1	5	2	3
σ_1 :	6	5	4	2	3	7	1
σ_2 :	2	5	4	6	3	1	7
id :	1	2	3	4	5	6	7

$23 \to 2^0 + 2^1 + 2^2 = 7$

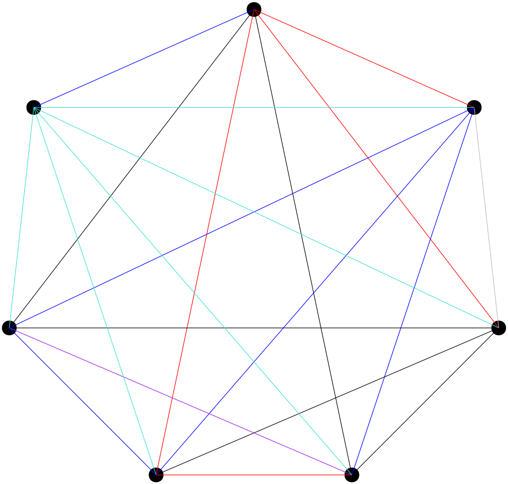
Pattern Graph

34

σ_0 :	7	4	6	1	5	2	3
σ_1 :	6	5	4	2	3	7	1
σ_2 :	2	5	4	6	3	1	7
id :	1	2	3	4	5	6	7

$34 \to 0$

Pattern Graph Example



With Multicolor Ramsey Theorem

$$n(a, b; 2) \leq R(\underbrace{b+2, b+2, \cdots, b+2}_{2^{a-1} \text{ copies}}) < \infty$$

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$$n(4, 1; 2) \leq R(3, 3, 3, 3, 3, 3, 3, 3) \leq 2.07689535... \times 10^{33}$$

$$n(4,1;2) = 13$$

Reduce Colors

Trivial Color

$$\begin{array}{cccccc}
\sigma_0 : & \cdots & a & \cdots & b & \cdots \\
\sigma_1 : & \cdots & a & \cdots & b & \cdots \\
\sigma_2 : & \cdots & a & \cdots & b & \cdots \\
\mathrm{id} : & \cdots & a & \cdots & b & \cdots
\end{array}$$

b, c : Red $\Rightarrow$ a : Blue

Reduce Colors

Irregular Colors

$$\begin{array}{llllll} \sigma_0 : & \cdots & b & \cdots & a & \cdots \\ \sigma_1 : & \cdots & b & \cdots & a & \cdots \\ \sigma_2 : & \cdots & b & \cdots & a & \cdots \\ \mathrm{id} : & \cdots & a & \cdots & b & \cdots \end{array}$$

- Case $b \neq n$ | a, n : Red $\Rightarrow$ b : Blue
- Case $b = n$ | Delete n.

Reduce Colors

Irregular Colors

σ_0 :	$\cdots$	b	$\cdots$	a	$\cdots$
σ_1 :	$\cdots$	b	$\cdots$	a	$\cdots$
σ_2 :	$\cdots$	b	$\cdots$	a	$\cdots$
id :	$\cdots$	a	$\cdots$	b	$\cdots$

- Case $b \neq n$ | a, n : Red $\Rightarrow$ b : Blue
- Case $b = n$ | Delete n.

Only three colors 1, 2, 4 remains!

Regular Colors

Transitivity

For $a < b < c$, if ab and bc have same color, then so is ac.

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No $K_{1,5}$

- If $1a$, $1b$, $1c$, $1d$, $1e$ have same color, then there is a monochromatic triangle.
- If an, bn, cn, dn, en have same color, then there is a monochromatic triangle.

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$n < 9$

If $n \geq 9$, then one of above always happens.

Final Result

Upper Bound

$$n(4,1;2) \leq 8 \text{ (Regular)} + 4 \text{ (Irregular)} + 1 = 13$$

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$$n(4,1;2) \leq 8 \text{ (Regular)} + 4 \text{ (Irregular)} + 1 = 13$$

Lower Bound

6	8	5	7	2	4	1	3
7	5	8	6	3	1	4	2
4	3	2	1	8	7	6	5
1	2	3	4	5	6	7	8

Final Result

Upper Bound

$$n(4,1;2) \leq 8 \ (\text{Regular}) + 4 \ (\text{Irregular}) + 1 = 13$$

Lower Bound

12	11	10	6	8	5	7	2	4	1	3	9
12	11	9	7	5	8	6	3	1	4	2	10
12	10	9	4	3	2	1	8	7	6	5	11
11	10	9	1	2	3	4	5	6	7	8	12

The end.