Domination numbers and the independence complexes in certain ternary graphs

Taehyun Eom

joint work with
Jinha Kim and Minki Kim

April 24, 2026

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$\mathbb{Z}_2$-Homology

- $(\mathbb{Z}_2\mathcal{F}_n, +) \simeq (\mathcal{P}(\mathcal{F}_n), \Delta)$.
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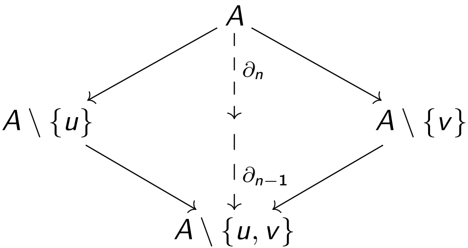

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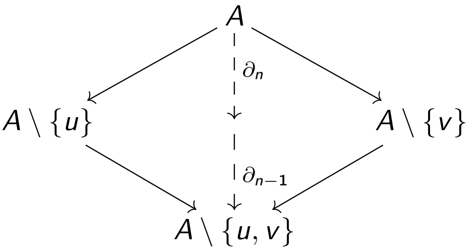


- $\partial_{n-1} \circ \partial_n = 0$.

$\mathbb{Z}$-Homology

If we fix an order for 'vertices' $\mathcal{F}_1$, we may define the boundary operator $\partial_{n+1} : \mathbb{Z}\mathcal{F}_{n+1} \to \mathbb{Z}\mathcal{F}_n$ from

$$\begin{aligned}\partial_{n+1}([\{u_0, u_1, \cdots, u_n\}]) = [\{&u_1, u_2, \cdots, u_n\}] \\ &- [\{u_0, u_2, \cdots, u_n\}] \\ &+ \cdots \\ &+ (-1)^n[\{u_0, u_1, \cdots, u_{n-1}\}]\end{aligned}$$

where $u_0 < u_1 < \cdots < u_n$ in the fixed order.

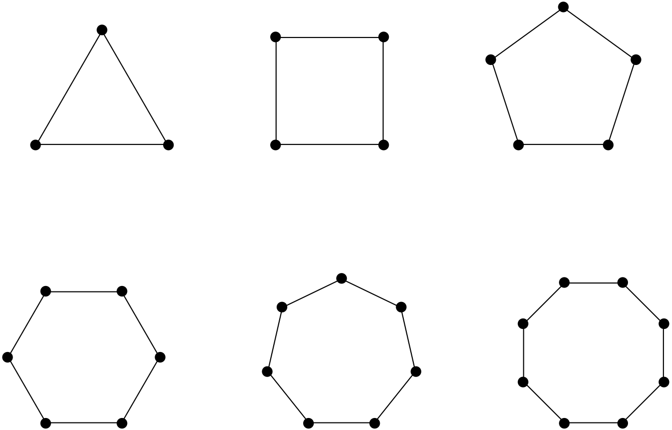
Homotopy

Actually,

- Every d-dimensional abstract simplicial complex can be embedded into the $\mathbb{R}^{2d+1}$.
- Every embeddings of an abstract simplicial complex are homeomorphic.

So, we can define the homotopy of abstract simplicial complexes, also.

Homotopy of the independence complex of a cycles



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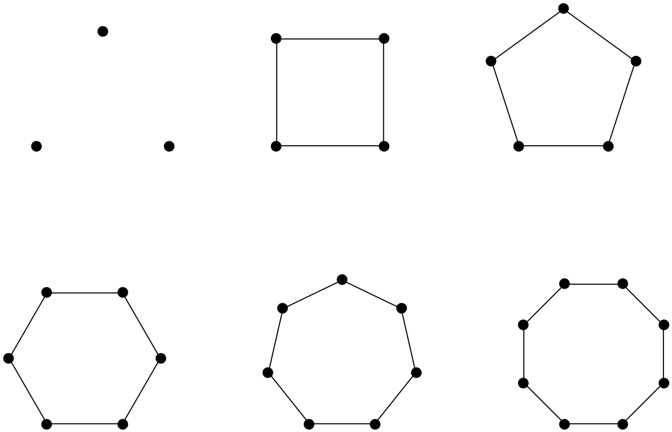

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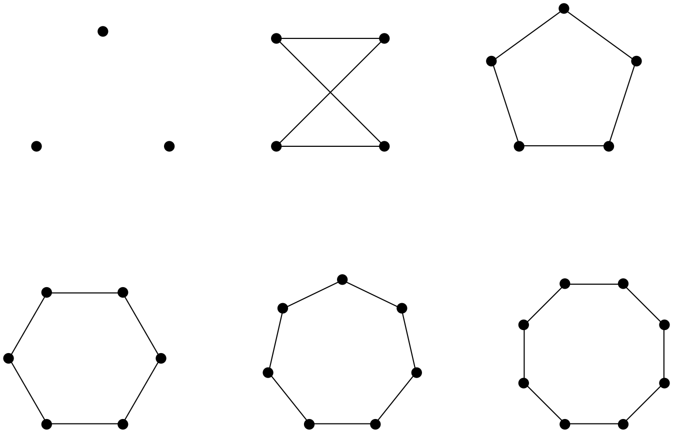

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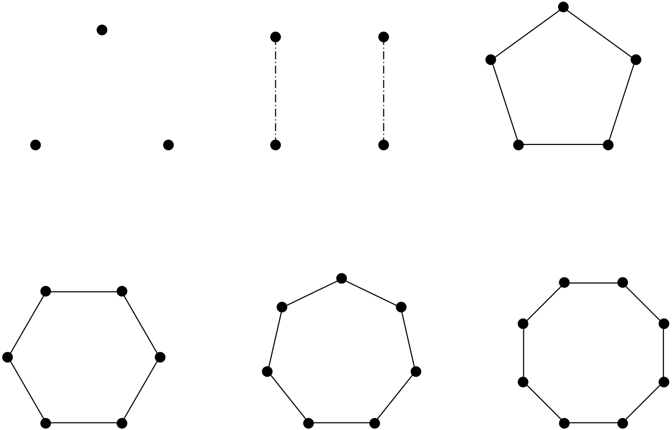

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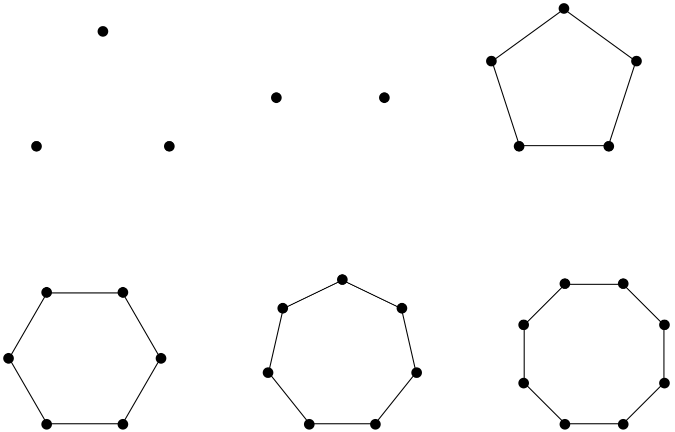

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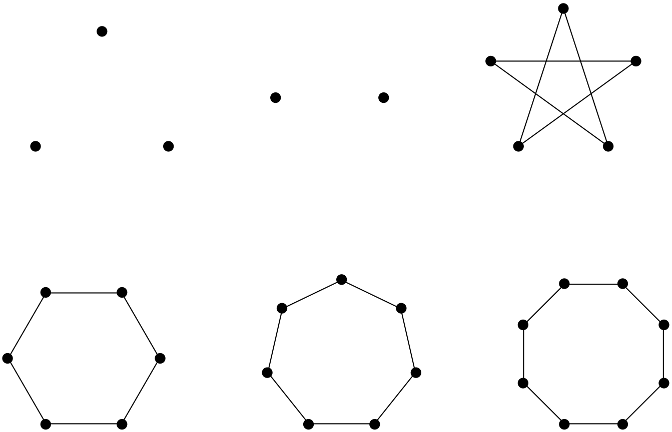

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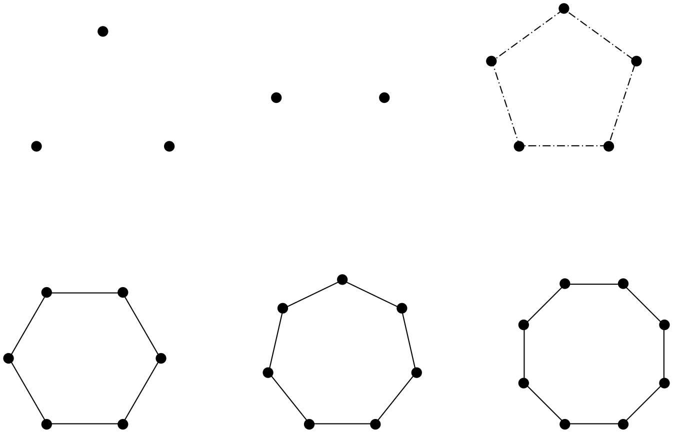

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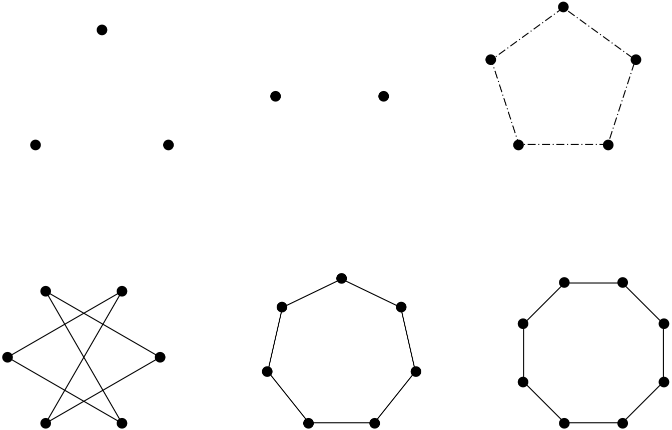

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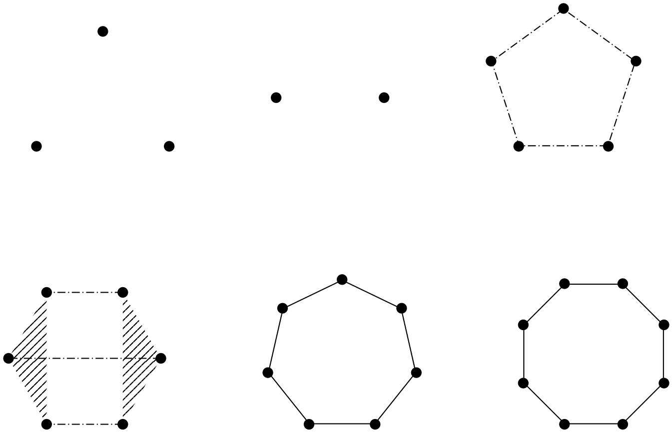

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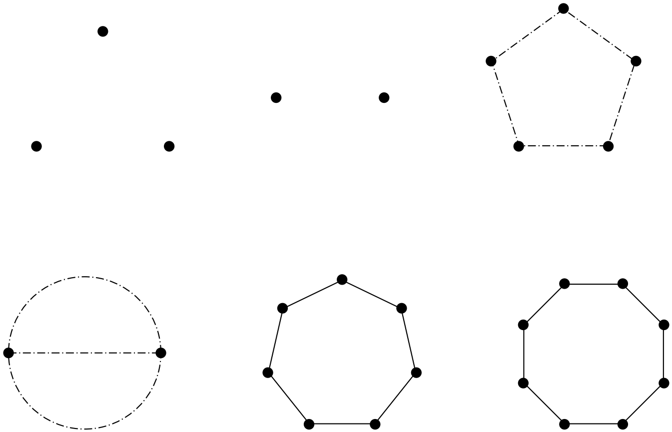

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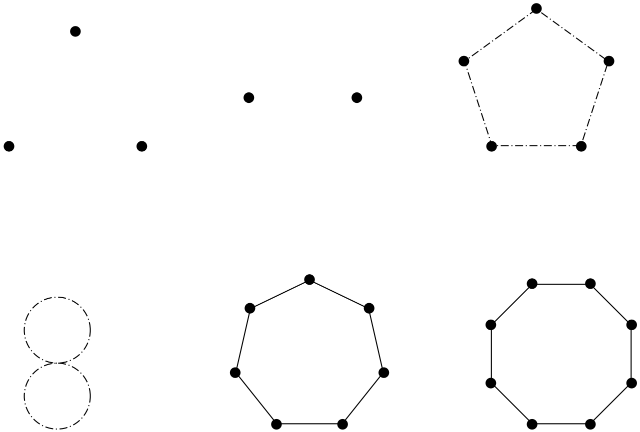

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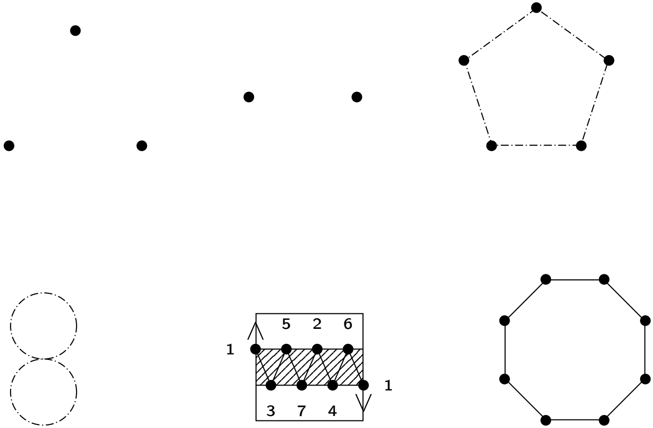

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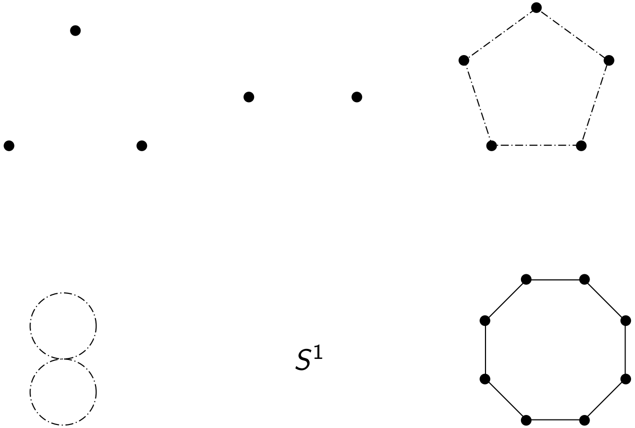

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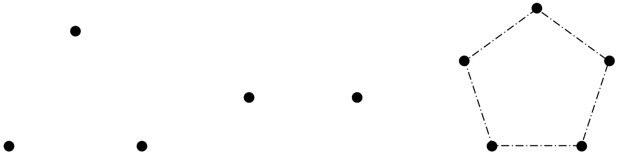

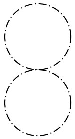

S^1

S^2

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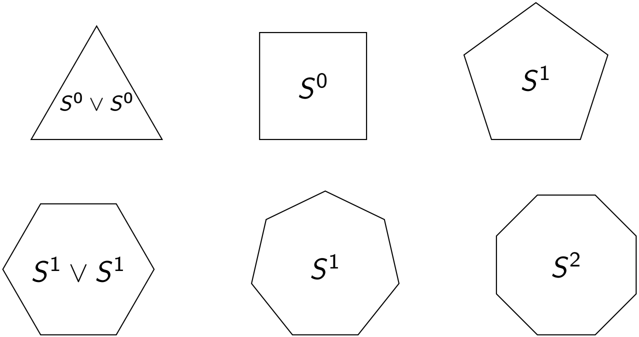

From Kalai, Meshulam and Engström

Ternary graph

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$\chi(I(G))$ for ternary graphs (Chudnovsky, Scott, Seymour and Spirkl, 2020)

For a ternary graph G, $\left|\sum_{A:\mathrm{indep}}(-1)^{|A|}\right| \leq 1$

Betti numbers of ternary graphs (Zhang and Wu, 2025, Arxiv 2020)

For a ternary graph G, the sum of reduced Betti number of $I(G)$ is at most 1.

Homotopy type of the ternary graph (J. Kim, 2022)

A graph G is ternary iff $I(H)$ is either contractible or homotopy equivalent to a sphere for every induced subgraph H of G.

Question

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For a ternary graph G with non-contractible $I(G)$, what is the dimension $d(G)$ of sphere $S^d \simeq I(G)$?

Homology vs Domination numbers

Domination numbers

- $\Gamma(S)$: the set of vertices dominated by S.
- $\gamma(G) = \min\{|S| : \Gamma(S) \supseteq V \setminus S\}$.
- $\gamma_0(G, A) = \min\{|S| : \Gamma(S) \supseteq A\}$.
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Aharoni and Chudnovsky, 2000

For $k < \frac{\gamma_0(G)}{2} - 1$, $\tilde{H}_k(I(G)) = 0$.

Aharaoni and Haxell, 2000

For an independent set A and $k < \gamma_0(G, A) - 1$, $\tilde{H}_k(I(G)) = 0$.

Aharoni, Berger and Ziv, 2002

For a chordal graph G and $k < \gamma(G) - 1$, $\tilde{H}_k(I(G)) = 0$.

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Known Result (Marietti and Testa, 2008)

For a forest F, if $I(F)$ is not contractible, then

$$d(F) = \gamma(F) - 1 = i(F) - 1$$

where $\gamma(F)$ is the domination number and $i(F)$ is the independent domination number of F.

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This result is also true for $C_5, C_8, \ldots$, but not for $C_4, C_7, \ldots$.

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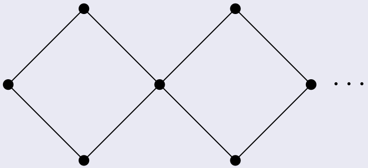
- For a (0,1)-ternary hypergraph H, if $I(H)$ is not contractible, then $d(H) = \gamma(H) - 1$.

With cycles of length 1 modulo 3

Connected C_4

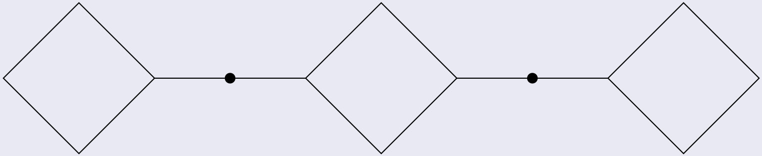


$3k - 1$ consecutive induced C_4.

- $I(\bullet) \simeq S^{2k-1}$
- $\gamma(\bullet) = i(\bullet) = 3k$

With cycles of length 1 modulo 3

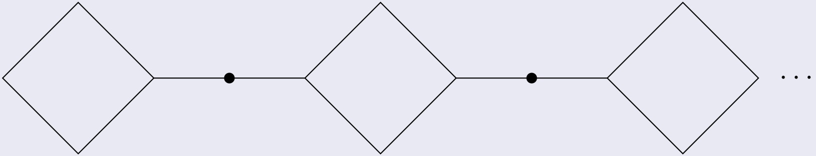
Connected C_4 by path of length 2



- $I(\bullet) \simeq S^3$
- $\gamma(\bullet) = i(\bullet) = 5$
- $\gamma(L(\bullet)) = 6$

With cycles of length 1 modulo 3

Connected C_4 by path of length 2



For the case with $3k$ induced C_4

- $I(\bullet) \simeq S^{4k-1}$
- $\gamma(\bullet) = i(\bullet) = \lceil \frac{9k}{2} \rceil$
- $\gamma(L(\bullet)) = 6k$

Further questions

Remaining problems

For ternary graphs,

- Is there any other parameter to represent the $d(G)$?
- $d(G) < \gamma(G)$?
- If $\gamma(G) \neq i(G)$, then $I(G)$ is contractible?

The end.