Fractional discrete Helly for pairs in a family of boxes

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joint work with
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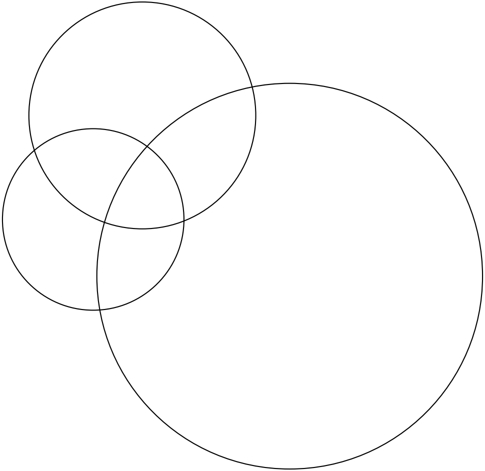
Helly's theorem

Helly's theorem for convex sets (1923)

For convex sets $X_1, \cdots .X_n \subseteq \mathbb{R}^d$, if every $d+1$ of them has nonempty intersection, then so is the total collection.



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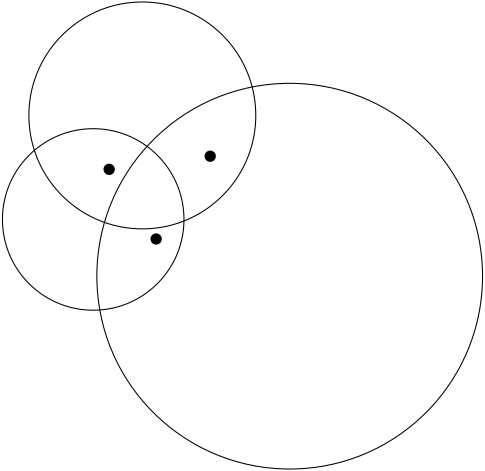

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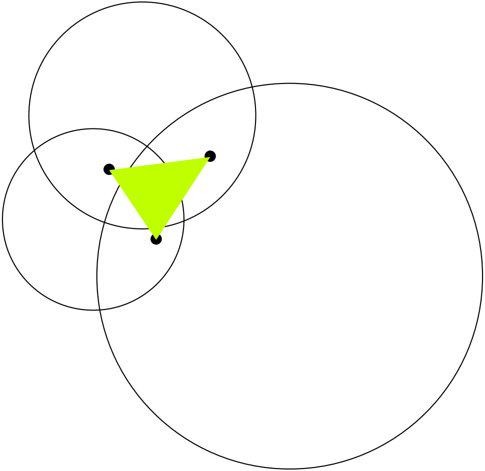

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This naturally comes from $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Fractional Helly theorem

Fractional Helly theorem (Katchalski and Liu, 1979)

For any $\alpha \in (0,1)$, there exists $\beta = \beta(\alpha, d) > 0$: for any family of convex sets $X_1, \cdots, X_n \subseteq \mathbb{R}^d$ that $\alpha\binom{n}{d+1}$ of $(d+1)$-tuples of them intersect, there exists an intersecting subfamily of size at least βn.

Restricted case

How Helly theorem will be changed if we restrict $\mathbb{R}^d$ into $S \subseteq \mathbb{R}^d$?

Restricted case

- S is a convex set in $\mathbb{R}^d$: For any family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, if $S \cap (\bigcap \mathcal{G}) \neq \emptyset$ for any $\mathcal{G} \in \binom{\mathcal{F}}{d+1}$, we have $S \cap (\bigcap \mathcal{F}) \neq \emptyset$.

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- $|S| = 1$: For any family $\mathcal{F}$ of sets, if $S \cap X \neq \emptyset$ for every $X \in \mathcal{F}$, then $S \cap (\bigcap \mathcal{F}) \neq \emptyset$.

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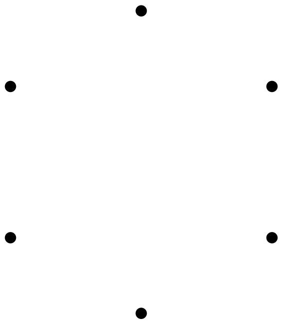
Can we do better than pigeonhole principle?

Restricted case



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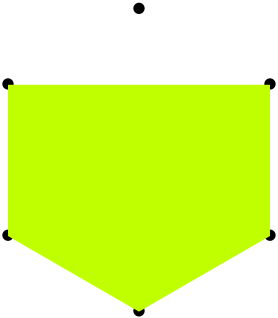


Discrete Helly theorem

Discrete Helly theorem for axis-parallel boxes (Halman, 2008)

Let $S \subseteq \mathbb{R}^d$ be a set of points. For axis-parallel boxes $B_1, \cdots, B_n \subseteq \mathbb{R}^d$, if every $2d$ of them intersects on S, then so is the total collection.

Fractional Helly theorems for axis-parallel boxes

Eckhoff, 1988

For axis-parallel boxes $B_1, \cdots, B_n \subseteq \mathbb{R}^d$ and a real number $\alpha \in (1 - \frac{1}{d}, 1]$, if at least $\alpha\binom{n}{2}$ pairs intersect, then there exists an intersecting subfamily of size at least $\left(1 - \sqrt{d(1-\alpha)}\right) n$.

Edwards and Soberón, 2025

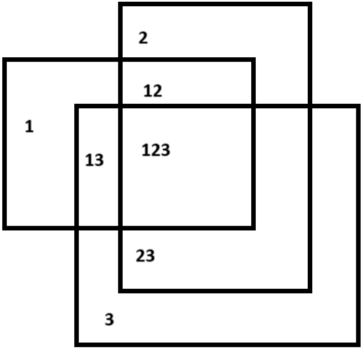
For any $\alpha \in (0, 1)$, there exists $\beta = \beta(\alpha, d)$ such that, for any point set $S \subseteq \mathbb{R}^d$ and axis-parallel boxes $B_1, \cdots, B_n \subseteq \mathbb{R}^d$, if $\alpha\binom{n}{d+1}$ of $(d+1)$-tuples intersects at S, then there exists a subfamily of size at least βn intersects at S.

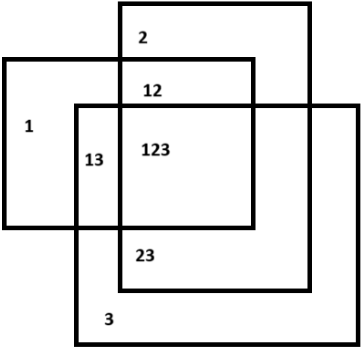
Fractional Helly theorems for axis-parallel boxes

E., Kim and Lee, 2025, *preprint*

- For any d, there exists $N = N(d)$ such that, for any point set $S \subseteq \mathbb{R}^d$ and axis-parallel boxes $B_1, \cdots, B_N \subseteq \mathbb{R}^d$, if every pair of boxes intersects at S, then there is a subfamily of size $d+1$ intersects at S.
- For any d, there exists a constant $c_d \in (0,1)$ and a function $\beta_d : (c_d, 1] \to (0,1]$ such that the following holds: for any $\alpha \in (c_d, 1]$, point set $S \subseteq \mathbb{R}^d$, and axis-parallel boxes $B_1, \cdots, B_n \subseteq \mathbb{R}^d$, if at least $\alpha\binom{n}{2}$ pairs of boxes intersect at S, then there exists a subfamily of size at least $\beta_d(\alpha)n$ intersects at S.

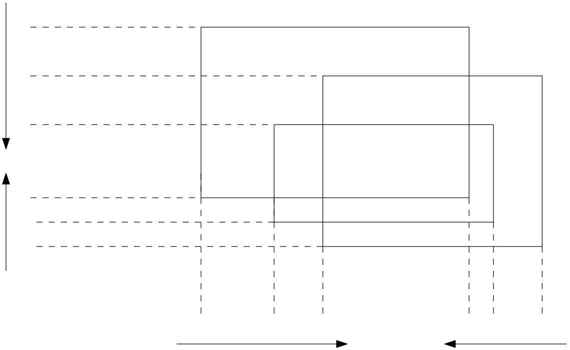
From geometry to algebra



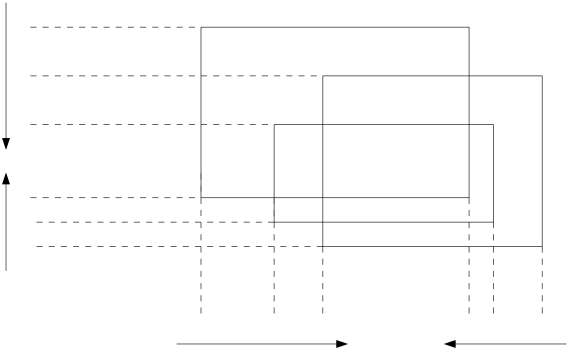


$N(d) > k$ is asking that, if there exists an k-boxes in $\mathbb{R}^d$ such that every pair is contained in some codeword length at most d.

From geometry to permutations



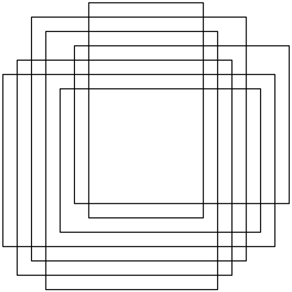
For n pairwise S-intersecting axis-parallel boxes in $\mathbb{R}^d$, we have $2d$ permutations in S_n.



For n pairwise S-intersecting axis-parallel boxes in $\mathbb{R}^d$, we have $2d$ permutations in S_n.
These permutations determine the intersection structure of boxes, so are codewords.

From geometry to permutations

7	4	6	1	2	5	3
6	5	4	2	3	7	1
2	5	4	6	3	1	7
1	2	3	4	5	6	7



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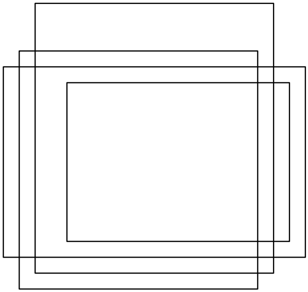

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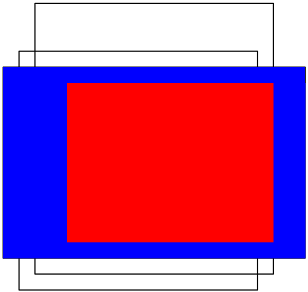

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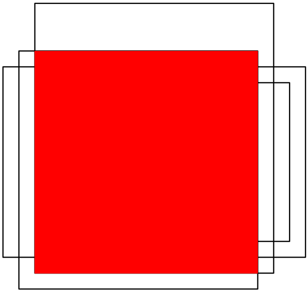

More permutation-wise problem

Question

If n is sufficiently large, then there exist two boxes such that their intersection is contained in $d-1$ boxes?

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Answer

There exists $n(a, b, p)$ such that if $n \geq n(a, b, p)$, then any a permutations in S_n have a proper choice of p red elements which gives at least b blue elements.

Geometric answer and combinatorial answer

$$N(d) \leq n(2d, d-1; 2) \leq R(\underbrace{d+1, d+1, \cdots, d+1}_{2^{2d-1}}) < \infty$$

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$$N(2) = 5 < n(4, 1, 2) = 13 \ll R(3, 3, 3, 3, 3, 3, 3, 3)$$

d-tuple intersecting pattern

E., Kim and Lee, 2025, *preprint*

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- There exists a constant C such that, for any point set $S \subseteq \mathbb{R}^d$ and axis-parallel boxes $B_1, \cdots, B_{Cd} \subseteq \mathbb{R}^d$, if every d-tuple of boxes intersects at S, then there is a subfamily of size $d+1$ intersects at S.

The end.