Domination numbers and homotopy in certain ternary graphs

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joint work with
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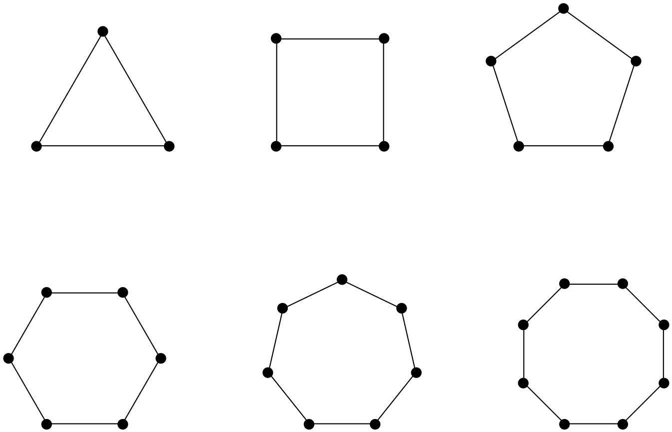
Independence complex, Clique complex, Nerve complex, ...

Independence complex of a cycle



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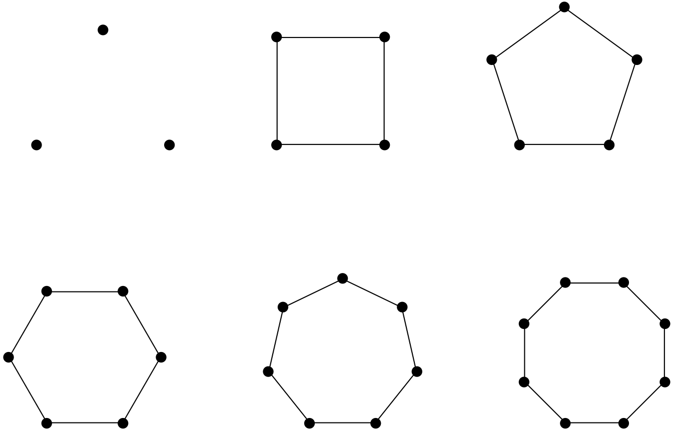

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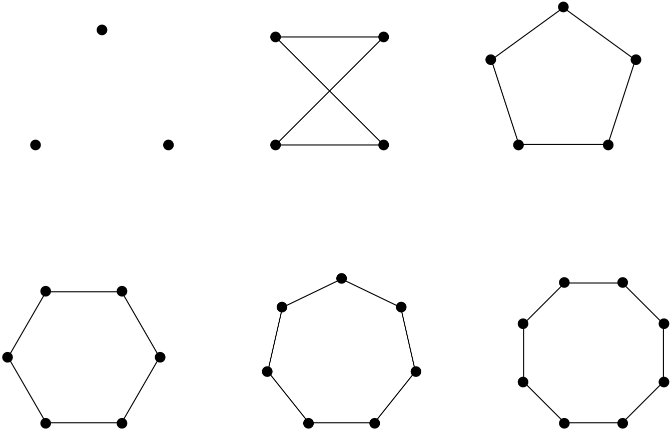

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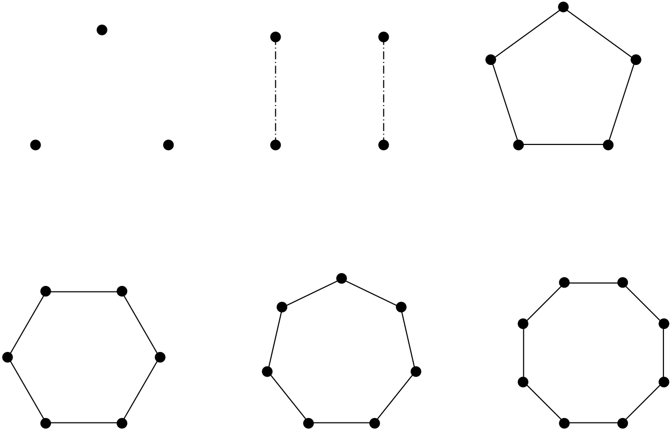

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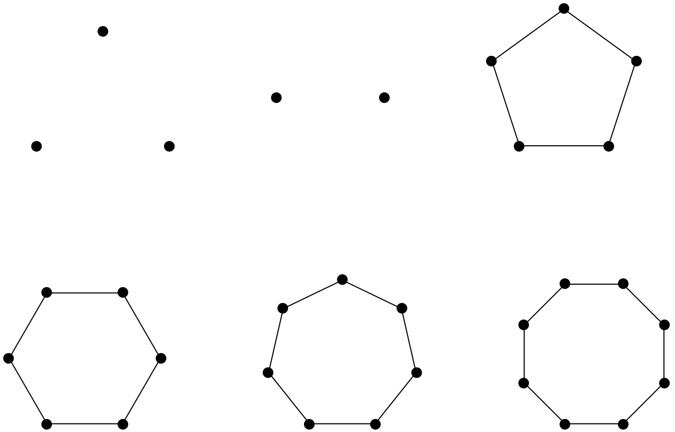

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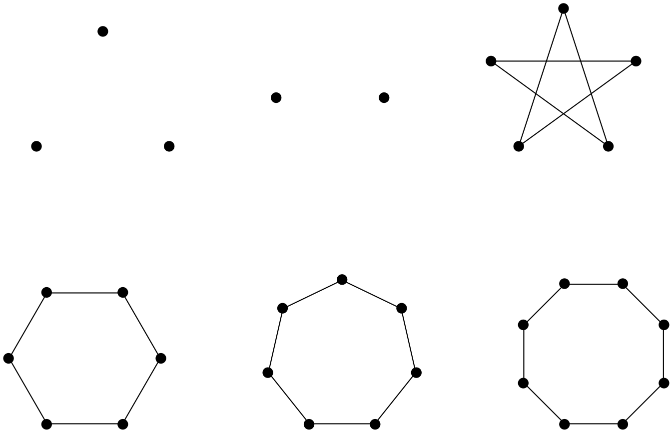

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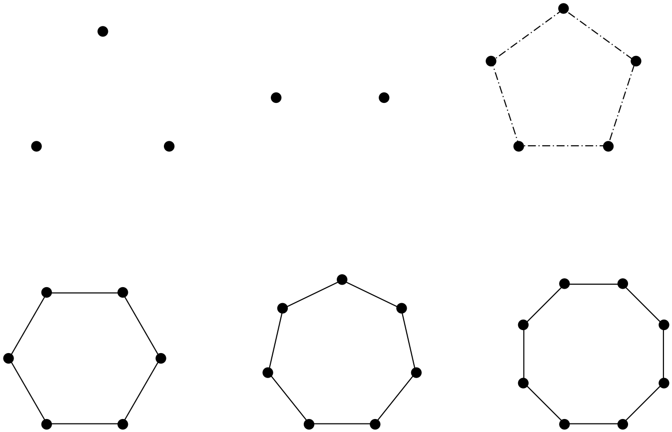

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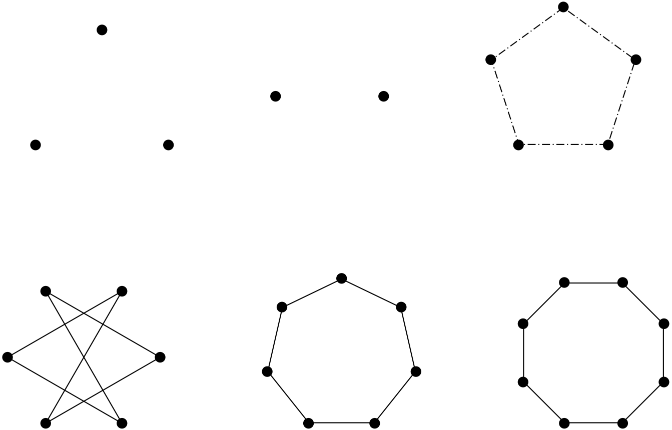

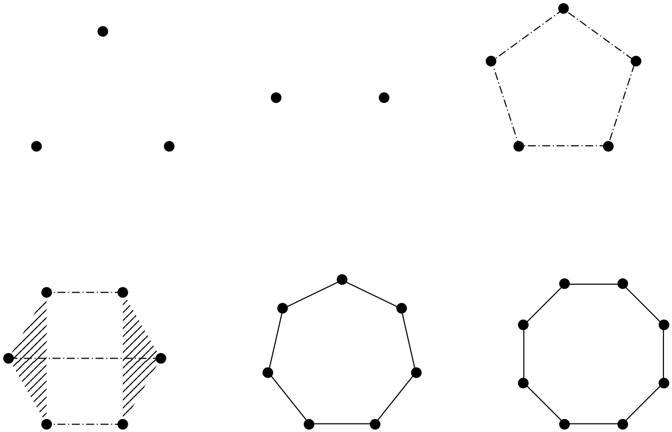

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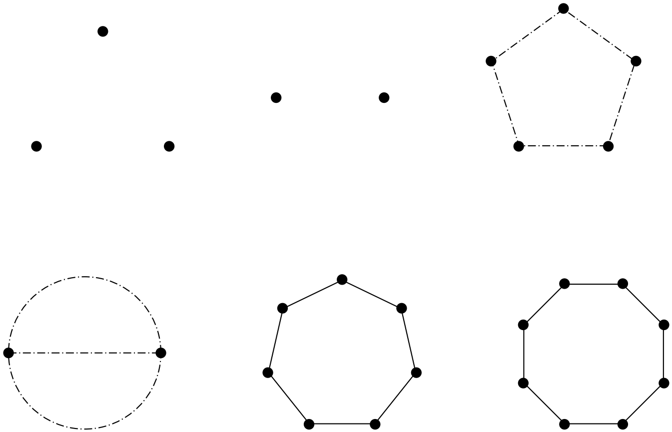

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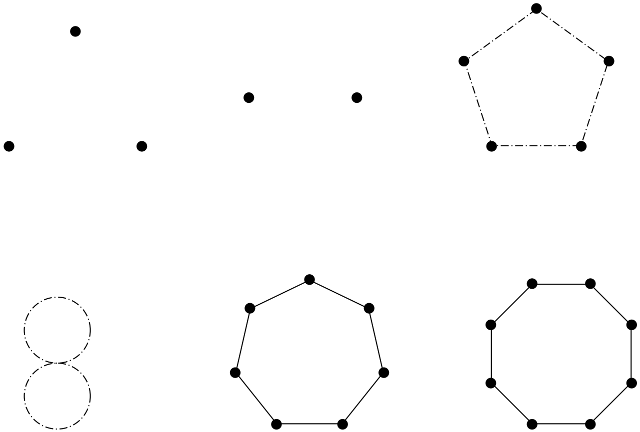

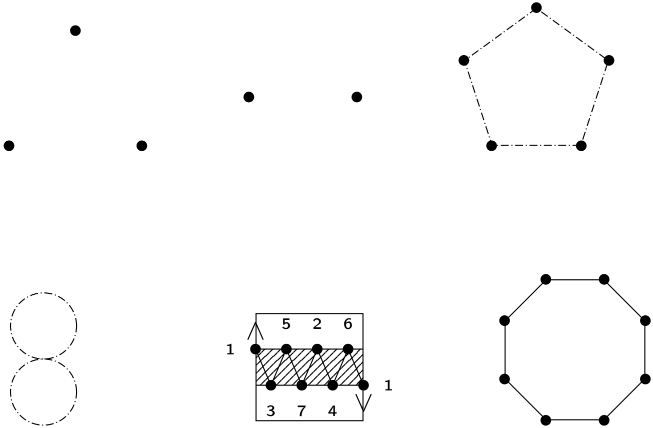
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1
1
3 7 4

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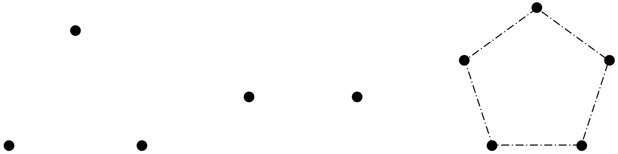
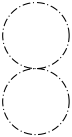
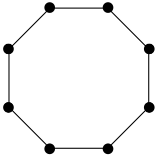
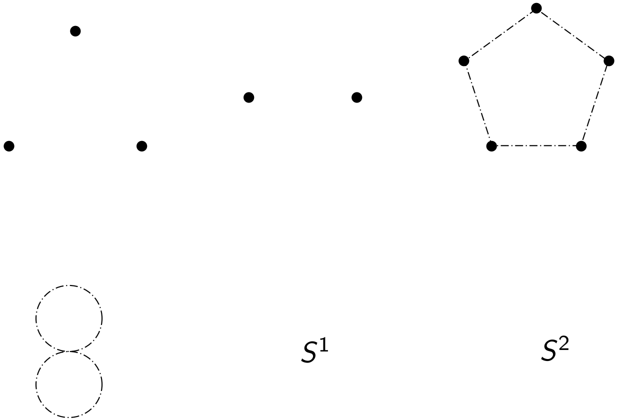




S^1




S^1
S^2

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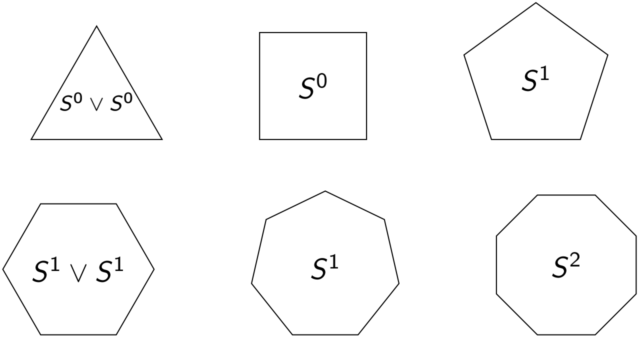

From Kalai, Meshulam and Engström

Ternary graph

A graph is **ternary** if it has no induced cycles of length 0 modulo 3.

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$\chi(I(G))$ for ternary graphs (Chudnovsky, Scott, Seymour and Spirkl, 2020)

For a ternary graph G, $\left|\sum_{A:\text{indep}}(-1)^{|A|}\right| \leq 1$

Betti numbers of ternary graphs (Zhang and Wu, 2025, Arxiv 2020)

For a ternary graph G, the sum of reduced Betti number of $I(G)$ is at most 1.

Homotopy type of the ternary graph (J. Kim, 2022)

A graph G is ternary iff $I(H)$ is either contractible or homotopy equivalent to a sphere for every induced subgraph H of G.

Question

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Known Result (Marietti and Testa, 2008)

For a forest F, if $I(F)$ is not contractible, then

$$d(F) = \gamma(F) - 1 = i(F) - 1$$

where $\gamma(F)$ is the domination number and $i(F)$ is the independent domination number of F.

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This result is also true for $C_5, C_8, \ldots$, but not for $C_4, C_7, \ldots$.

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- For a (0,1)-ternary hypergraph H, if $I(H)$ is not contractible, then $d(H) = \gamma(H) - 1$.

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$$d(G) \leq \gamma(L(G)) - 1 \leq \gamma(G) - 1 \leq i(G) - 1 \leq d(G)$$

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Graph point of view, Hà and Woodroofe, 2014

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E., Kim and Kim

For a $(0,1)$-ternary graph G, $\gamma(L(G)) \leq \gamma(G)$

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- For a connected $(0,1)$-ternary graph G, there exists a vertex v such that $i(G - v) \geq i(G)$.
- For a $(0,1)$-ternary graph G, if $I(G)$ is not contractible, then $i(G) - 1 \leq d(G)$.

Structure of $(0,1)$-ternary graphs. Part 0

Without inducedness

A graph G is $(0,1)$-ternary if and only if G has no cycles of length 0 or 1 modulo 3.

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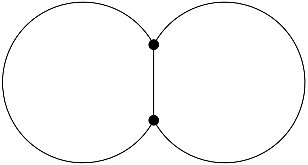

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A $(0, 1)$-ternary graph is K_4-(topological)-minor free.

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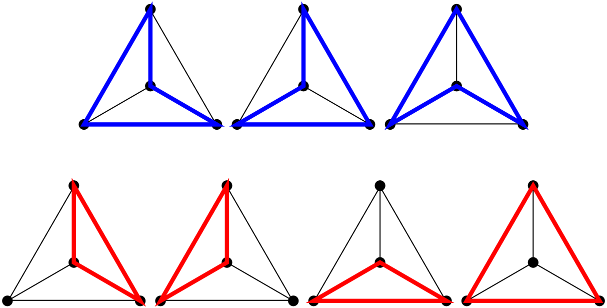

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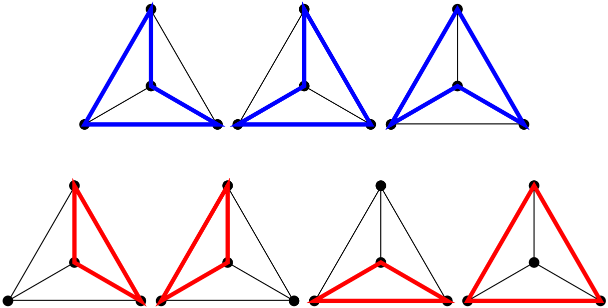

This is generally true for any graph G such that every cycle of length $k(\neq 0)$ modulo m

Structure of $(0,1)$-ternary graphs. Part 2

2-connected and ear-decomposition

For a 2-connected $(0,1)$-ternary graph G and its ear-decomposition sequence $G_0 \leq G_1 \leq \cdots \leq G_k = G$, let u, v be two endpoints of the last ear E.

- Every path connecting u and v has length 1 modulo 3.
- u and v are lying on the same ear in G_{k-1}.

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Since G is 2-connected, there exists two paths P, P' connecting u and v in G_{k-1}. Since $P \cup E$, $P' \cup E$ and $P \cup P'$ have length 2 modulo 3, P, P' and E have length 1 modulo 3.

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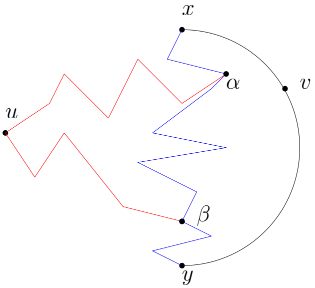
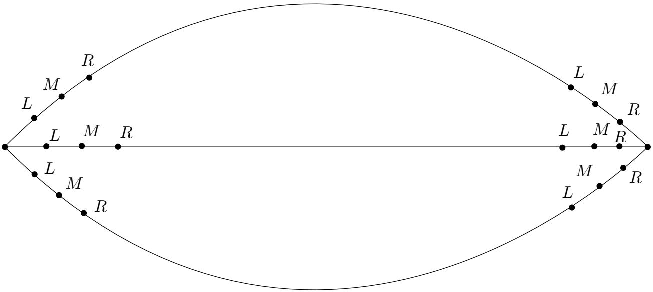


Structure of minimum independent dominating sets



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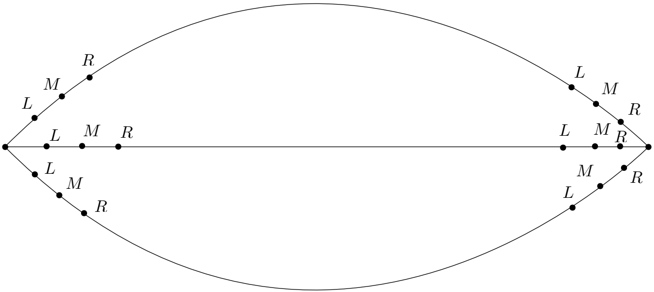

The number of vertices is $3k + 2$, and its independent domination number is $k + 1$.

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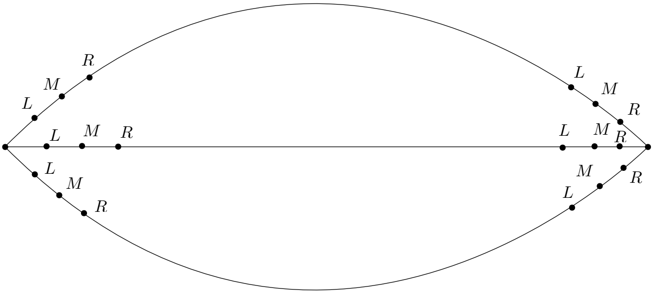

The number of vertices is $3k+2$, and its independent domination number is $k+1$.

- If both endpoints are not in a minimum size independent dominating set, then one path has pattern $L \cdots L(LR)R \cdots R$ and others have $M \cdots M$.

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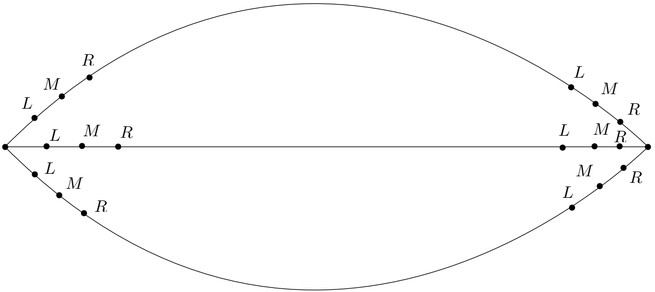

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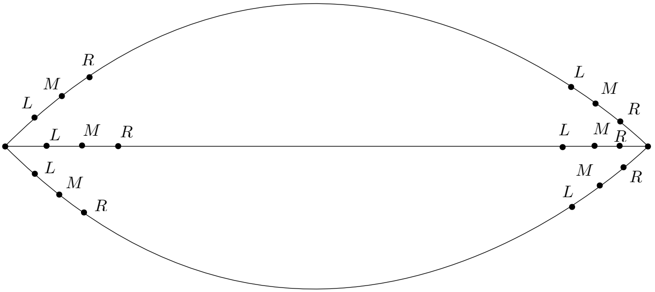

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- If the right endpoint is in a minimum size independent domination set, then there is a path with pattern $L \cdots L$ and others have $M \cdots ML \cdots L$.

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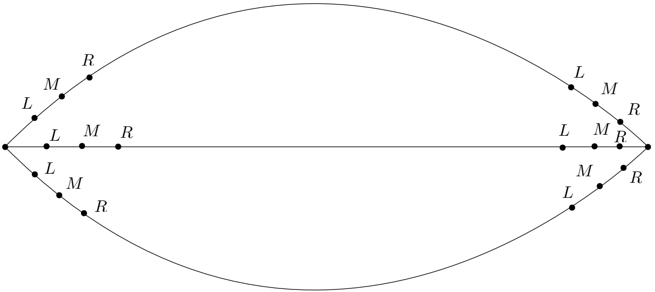

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- Both endpoints cannot be included in a minimum size independent dominating set simultaneously.

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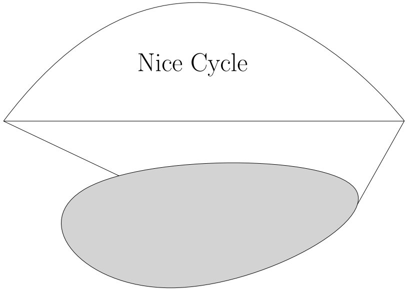

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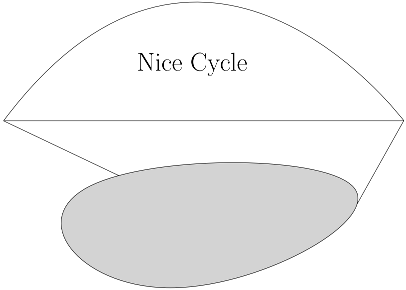

Independent domination of 2-connected (0,1)-ternary graphs

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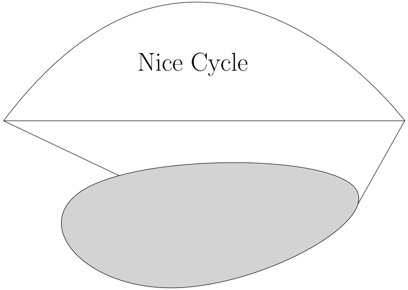

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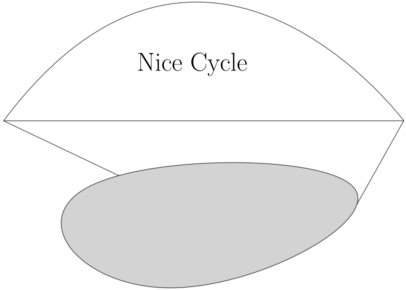

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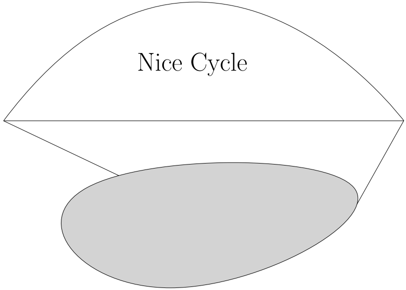

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- For any $v \in V(G)$, $i(G - v) \geq i(G)$.

Structure of minimum independent dominating sets

Existence of a special vertex

For a connected (0,1)-ternary graph G, there exists a vertex v with $i(G - v) \geq i(G)$.

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For (0,1)-ternary graph which is not 2-connected, we may choose v as the cut vertex of a leaf block of the block-cut tree of G.

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Existence of a special vertex

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Independent domination number vs Sphere dimension

For a (0,1)-ternary graph G, if $I(G)$ is not contractible, then

$$i(G) - 1 \leq d(G).$$

This uses a Mayer-Vietoris argument based on $G - v$ and $G - N(v)$ to prove $\tilde{H}_s(I(G)) = 0$ for $s \leq i(G) - 2$.

Edge domination number

Domination number vs Edge domination number

For a (0,1)-ternary graph G,

$$\gamma(L(G)) \leq \gamma(G).$$

Sketch of the proof) Generally, for a (0,1)-ternary graph G and a minimum size dominating set W, there exists $W' \subseteq W$ and edges $e_1, \cdots, e_{|W'|}$ where each e_i attaches to a vertex in W' such that $W \setminus W'$ with $e_1, \cdots, e_{|W'|}$ dominates edges.

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- $\gamma(L(G)) \leq \gamma(L(G) - \{e_1, \cdots, e_{|W'|}\}) + |W'|$.

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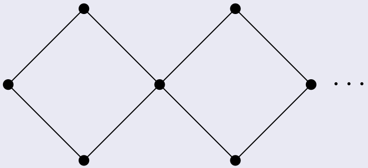
- $\gamma(L(G) - \{e_1, \cdots, e_{|W'|}\}) \leq \gamma(G') \leq |W| - |W'|$.
- $\gamma(L(G)) \leq |W| - |W'| + |W'| = |W| = \gamma(G)$.

With cycles of length 1 modulo 3

Connected C_4

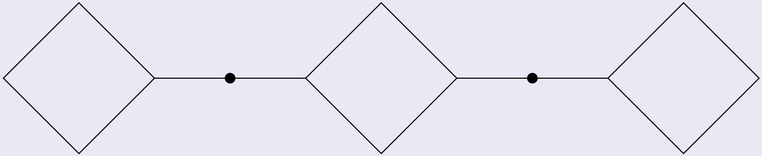


$3k-1$ consecutive induced C_4.

- $I(\bullet) \simeq S^{2k-1}$
- $\gamma(\bullet) = i(\bullet) = 3k$

With cycles of length 1 modulo 3

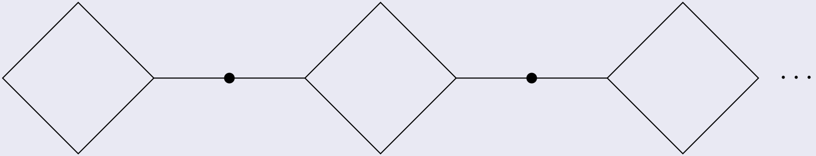
Connected C_4 by path of length 2



- $I(\bullet) \simeq S^3$
- $\gamma(\bullet) = i(\bullet) = 5$
- $\gamma(L(\bullet)) = 6$

With cycles of length 1 modulo 3

Connected C_4 by path of length 2



For the case with $3k$ induced C_4

- $I(\bullet) \simeq S^{4k-1}$
- $\gamma(\bullet) = i(\bullet) = \lceil \frac{9k}{2} \rceil$
- $\gamma(L(\bullet)) = 6k$

Further questions

Remaining problems

- Is there any other parameter to represent the $d(G)$, which explains cases with length 1 modulo 3 cycles?
- $\frac{2}{3}\gamma(G) \leq d(G) + 1 \leq \gamma(G)$?
- If $\gamma(G) \neq i(G)$, then $I(G)$ is contractible?

The end.