Forbidden sub-co-walks

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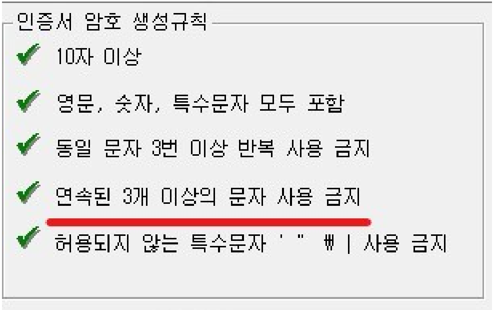
Rules for “safe” passwords

인증서 암호 생성규칙

- ✔ 10자 이상
- ✔ 영문, 숫자, 특수문자 모두 포함
- ✔ 동일 문자 3번 이상 반복 사용 금지
- ✔ 연속된 3개 이상의 문자 사용 금지
- ✔ 허용되지 않는 특수문자 ' " ₩ | 사용 금지

Rules for “safe” passwords



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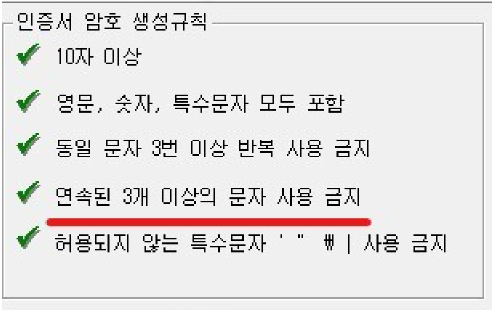

Is this rule really good for safety?

How to count?

$$
\begin{tikzcd}
& \cdots \mathrm{aa} \\
\cdots \mathrm{aa} \arrow[ur, "+\mathrm{a}"] \arrow[r, "+\mathrm{b}"] \arrow[dr, "+\mathrm{c}"'] & \cdots \mathrm{ab} \\
& \cdots \mathrm{ac}
\end{tikzcd}
$$

$$\vdots$$

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\cdots \mathrm{aa} \arrow[ur, "+\mathrm{a}"] \arrow[r, "+\mathrm{b}"] \arrow[dr, "+\mathrm{c}"'] & \cdots \mathrm{ab} \\
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Create a De Brujin-like graph $\rightarrow$ Count the number of walks.

In the point of view of alphabets

$$\begin{tikzcd}
 & \mathrm{a} \\
\mathrm{a} \arrow[ur] \arrow[r, dashed] \arrow[dr] & \mathrm{b} \\
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 & \vdots
\end{tikzcd}$$

In the point of view of alphabets

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\begin{tikzcd}
 & a \\
a \arrow[ur] \arrow[r, dashed] \arrow[dr] & b \\
 & c \\
 & \vdots
\end{tikzcd}
\qquad
\begin{tikzcd}
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b \arrow[ur] \arrow[r] \arrow[dr, dashed] & b \\
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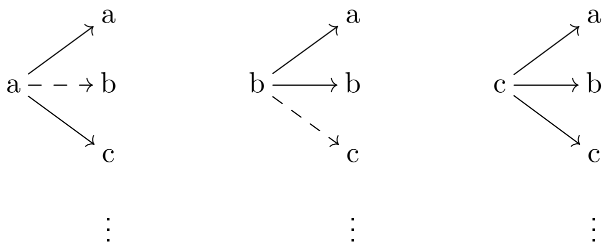

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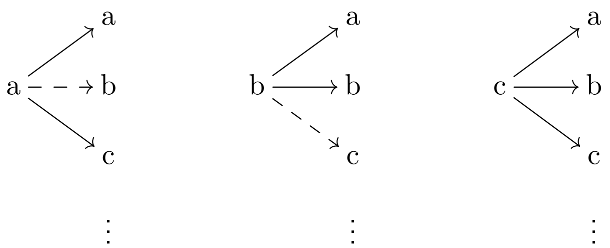
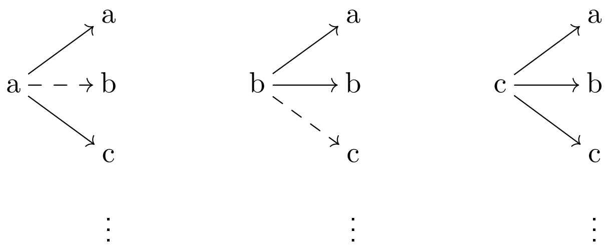


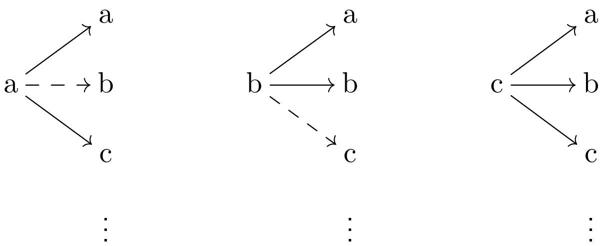
Count the number of vertex sequences such that

In the point of view of alphabets



Count the number of vertex sequences such that no consecutive co-edge appears

In the point of view of alphabets



Count the number of vertex sequences such that every *sub-co-walk* has length at most 1

Basic Definitions

- D : A simple directed graph allows loops on an n-vertex set V
  $\Leftrightarrow n \times n$ binary matrix
- J_n : $n \times n$ matrix of ones
- 1_n : $n \times 1$ vector of ones
- $\theta_m^{(k)}(D)$: The number of vertex sequences $v_0 \cdots v_m$ such that every sub-co-walk has length at most k
- $\mathcal{L}^{(k)}(D)$: kth order line digraph of D with a vertex set V^{k+1}
- ρ : The spectral radius of a matrix

Basic Properties

- $\mathcal{L}^{(k)}(J_n)$: $(k+1)$-dimensional De Brujin graph of n symbols
- $\mathcal{L}^{(0)}(D) = D$
- $\theta_m^{(0)}(D)$: The number of length m walks of D
- $1_{n^{k+1}}^T \left[\mathcal{L}^{(k)}(D)\right]^m 1_{n^{k+1}} = \theta_{m+k}^{(0)}(D)$ if $m \geq 1$

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How to compute $\theta_m^{(k)}(D)$?

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Note that 2) step is same as deleting arcs from $\mathcal{L}^{(k)}(J_n)$ when $\alpha \to \beta$ is an arc of $\mathcal{L}^{(k)}(J_n - D)$.

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3) Let such digraph be $\delta^{(k)}(D)$.
4) Then, $1^T_{n^{k+1}} \left[\delta^{(k)}(D)\right]^m 1_{n^{k+1}} = \theta^{(k)}_{m+k}(D)$ for $m \geq 1$.

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$$\delta^{(k)}(D) = \mathcal{L}^{(k)}(J_n) - \mathcal{L}^{(k)}(J_n - D).$$

Perron-Frobenius theorem

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For any irreducible non-negative matrix A, $\rho(A)$ is a simple eigenvalue of A and it corresponds to a positive eigenvector.

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For reducible non-negative matrix, its spectrum is union of spectra of submatrices based on irreducible components, i.e. strongly connected components in underlying directed graph.

Perron-Frobenius theorem

Corollary

For an irreducible non-negative matrix A and its positive eigenvector x of $\rho(A)$,

$$\left[\left(\min_i \frac{w_i}{x_i}\right) v^T x\right] \rho(A)^m \leq v^T A^m w \leq \left[\left(\max_i \frac{w_i}{x_i}\right) v^T x\right] \rho(A)^m$$

for every non-negative vector v, w.

Perron-Frobenius theorem

Corollary

For any non-negative $n \times n$ matrix A, there exists a constant $c \geq 1$ and a polynomial p such that

$$c\rho(A)^m \leq 1_n^T A^m 1_n \leq p(m)\rho(A)^m$$

for every $m \geq n$. Hence,

$$\lim_{m\to\infty} \left[1_n^T A^m 1_n\right]^{1/m} = \rho(A).$$

Here, if A is irreducible, then polynomial p be a constant polynomial. This gives

$$\rho(\delta^{(k)}(D)) = \lim_{m\to\infty} \left[\theta_{m+k}^{(k)}(D)\right]^{\frac{1}{m}} = \lim_{m\to\infty} \left[\theta_m^{(k)}(D)\right]^{\frac{1}{m}}$$

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Corollary

For a non-negative matrix A,

- if μ is the minimum value among non-zero entries, $\rho(A) < \mu$ implies $\rho(A) = 0$. In particular, $\rho(D) \neq 0$ implies $\rho(D) \geq 1$ for a binary matrix D.
- $\rho(A) \neq 0$ if and only if its underlying directed graph does not contains a cycle.

Additional Properties

Proposition

- If $\delta^{(k)}(D)$ is irreducible, then $\delta^{(k+1)}(D)$ is also irreducible.
- If D has a source or sink, then for every k, $\delta^{(k)}(D)$ cannot be irreducible.
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This also proves that $\left[\delta^{(k)}(D)\right]^m$ is positive for $k \geq 1$ and $m \geq k+3$ if D has no source nor sink.

Additional Properties

Proposition

For $k \geq 1$, let $q = \lfloor \frac{m}{k+1} \rfloor$, $r = m - (k+1)q$. Then,

$$\theta_m^{(k)}(D) \geq [\#Arc(D)]^q n^{(k-1)q+r+1} \geq [\#Arc(D)]^q n^{m\left(1-\frac{2}{k+1}\right)+1}.$$

Note that $\#Arc(D) = 1_n^T D 1_n = \theta_1^{(0)}(D)$.

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Proof. It is easy to check that every vertex sequence in

$$V \times V^{k-1} \times Arc(D) \times V^{k-1} \times Arc(D) \times \cdots \times V^{k-1} \times Arc(D) \times V^r$$

is valid sequence when consider $\theta_m^{(k)}(D)$. □

Additional Properties

Corollary

Suppose $D \neq 0$.

- $\rho(\delta^{(k)}(D)) \geq n^{1-\frac{2}{k+1}}$ if $k \geq 1$.
- $\lim_{k\to\infty} \rho(\delta^{(k)}(D)) = n$.
- For any function $f : \mathbb{N} \to \mathbb{N}$ such that $\lim_{m\to\infty} f(m) = \infty$, we have

$$\lim_{m\to\infty} [\theta_m^{(f(m))}(D)]^{\frac{1}{m}} = n.$$

- For any $\tau \in (0, 1)$,

$$\lim_{m\to\infty} [\theta_m^{(\tau m)}(D)]^{\frac{1}{m}} = n.$$

Special Case

Proposition

If D and D' are n-vertex outerdegree d-regular,

$$\theta_m^{(k)}(D) = \theta_m^{(k)}(D')$$

for every $k, m \geq 0$ and hence, $\rho(\delta^{(k)}(D)) = \rho(\delta^{(k)}(D'))$.

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Proof 1. Assume that D and D' have common vertex set V.
Define $f_{v,D} : \{(0,1), \cdots, (0,d), (1,1), \cdots, (1,n-d)\} \to V$ as

- $f_{v,D}((0,i))$ is ith vertex among v's neighbors.
- $f_{v,D}((1,i))$ is ith vertex among v's non-neighbors.

Similarly, define $f_{v,D'}$. These functions are well-defined since D, D' are outerdegree d-regular.

Special Case

Consider the following bijection from vertex sequences of D to vertex sequences of D'

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$$\begin{array}{ccccc} v_0 & v_1 & v_2 & \cdots & v_m \\ w_0 & w_1 & w_2 & \cdots & w_m \end{array}$$

This bijection preserves sub-co-walk structure, so we have

$$\theta_m^{(k)}(D) = \theta_m^{(k)}(D'),$$

and hence, we have

$$\rho(\delta^{(k)}(D)) = \lim_{m\to\infty} \theta_m^{(k)}(D)^{\frac{1}{m}} = \lim_{m\to\infty} \theta_m^{(k)}(D')^{\frac{1}{m}} = \rho(\delta^{(k)}(D'))$$

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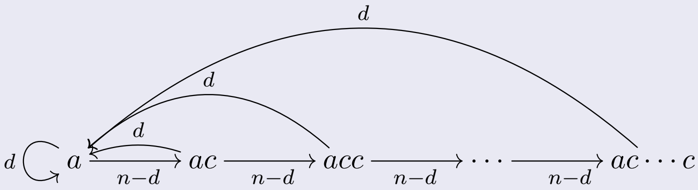
Proof 2.



Special Case

Thus, we have

$$\theta_m^{(k)}(D) = 1_{k+1}^T \begin{bmatrix} d & d & \cdots & d \\ n-d & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & n-d & 0 \end{bmatrix}^m \begin{bmatrix} n \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \theta_m^{(k)}(D')$$

and

$$\rho(\delta^{(k)}(D)) = \rho(\delta^{(k)}(D')) = \rho\left(\begin{bmatrix} d & d & \cdots & d \\ n-d & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & n-d & 0 \end{bmatrix}\right)$$

Reduction

In $n^{k+1} \times n^{k+1}$ matrix $\delta^{(k)}(D)$, there exists many redundancies. As similar as regular case, when we add a vertex to given vertex sequence, it is enough to consider only after the last arc appears.

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- $\widetilde{\Theta}_k$: the set of co-walks w of length at most k such that $[^\wedge w] \neq \emptyset$.

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In $n^{k+1} \times n^{k+1}$ matrix $\delta^{(k)}(D)$, there exists many redundancies. As similar as regular case, when we add a vertex to given vertex sequence, it is enough to consider only after the last arc appears.

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Note that $[^\wedge w] = \emptyset$ if and only if w starts with a source and the length is less than k.

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  1) Choose a co-walk w in $\widetilde{\Theta}_k$ such that the length is less than k, and a vertex v.
  2) If wv is a co-walk, draw an arc $w \to wv$.
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Proposition

- $[\delta^{(k)}(D)]P = P[\widetilde{\delta}^{(k)}(D)]$
- $1^T_{n^{k+1}} P[\widetilde{\delta}^{(k)}(D)]^m 1_{\widetilde{\Theta}_k} = \theta^{(k)}_{m+k}(D)$ for $m \geq 1$.
- $\rho(\delta^{(k)}(D)) = \rho(\widetilde{\delta}^{(k)}(D))$

This is usual process using equitable partitions.

Approximation

For $v, v' \in V$, consider a conditional probability

$$p_{vv'} = P(wv' \text{ is not a co-walk} | w = v_0 \cdots v_{k-1} v)$$

and the $n \times n$ matrix $\widehat{\delta}^{(k)}(D)$ consists of $p_{vv'}$.

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Note that if $vv' \in Arc(D)$, then $p_{vv'} = 1$ and if $vv' \notin Arc(D)$,

$$p_{vv'} = 1 - \frac{\#\text{length } k \text{ co-walk ends with } v}{n^k} = 1 - \frac{1_n^T [J_n - D]^k e_v}{n^k}$$

where e_v is the characteristic vector for v.

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3) Then, we get
$$\widehat{\delta}^{(k)}(D) = D+(I_n-Q^{(k)}(D))(J_n-D) = J_n-Q^{(k)}(D)[J_n-D].$$

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3) Let $n_{vv'}$ be the number of 1's in $[-v, -v']$.
4) Generate a matrix consists of $\frac{n_{vv'}}{n^k}$.

Approximation

Miscellaneous

- $\lim_{k\to\infty} Q^{(k)}(D) = 0$ if $D \neq 0$.
- $\lim_{k\to\infty} \rho(\widehat{\delta}^{(k)}(D)) = n$.
- For $n \times n$ binary matrices with exactly one 1, $E[\rho] = \frac{1}{n}$.
- For $n \times n$ binary matrices with exactly two 1's, $E[\rho] = \frac{2}{n}$.
- For $n \times n$ binary matrices with exactly three 1's,

$$E[\rho] = \frac{3}{n}\left(1 - \frac{3-\sqrt{5}}{(n+1)(n^2-2)}\right).$$

About construction

- Is it possible to define η properly so

$$\rho(\eta^k(D)) = \rho(\eta(\eta(\cdots\eta(D)\cdots))) = \rho(\delta^{(k)}(D))?$$

- Is there any natural algebraic operator $\circ$ such that

$$[\delta^{(k)}(D)] \circ D = [\delta^{(k+1)}(D)]?$$

- Is it possible to define theses concepts for multigraphs?

About values

- For given D, is it possible to find a concrete bound $p_k(m)$ satisfying
$$\frac{1_{n^{k+1}}^T[\delta^{(k)}(D)]^m 1_{n^{k+1}}}{\rho(\delta^{(k)}(D))^m} \leq p_k(m)$$
which satisfies $[p_{f(m)}(m)]^{1/m} \to 1$ for every, or certain conditioned, f with $f \to \infty$. Is it possible to find such p_k does not depends on k?
- Is there any relation between $\rho(\delta^{(k)}(D))$ and
$$\rho\left(\begin{bmatrix} d & d & \cdots & d \\ n-d & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & n-d & 0 \end{bmatrix}\right)$$
where d is the mean outerdegree.

About values

- We have $\rho(\delta^{(k)}(D)) \geq n^{1-\frac{2}{k+1}}$ as a natural bound. Then, if we consider some normalization such as

$$\left[\frac{\rho(\delta^{(k)}(D))}{n^{1-\frac{2}{k+1}}}\right]^{\frac{k+1}{2}},$$

what can we say about its limit?

About reductions

- Find conditions for the irreducibility of $\widetilde{\delta}^{(k)}(D)$.
- Is there any remarkable facts for the ranks of $\delta^{(k)}(D)$ and $\widetilde{\delta}^{(k)}(D)$?
- Is there any better reduction for certain types of D, which has weaker condition than regularity?

About approximation

- Is $\widehat{\delta}^{(k)}(D)$ really good approximation for $\delta^{(k)}(D)$ to compute ρ?
- For $n \times n$ binary matrices with exactly k 1's, is the following satisfied?
$$nE[\rho] \to k \text{ as } n \to \infty$$
How about matrices with some restrictions like $[-v, -v']$ has? For example, each row has at most one 1, or only certain columns are possible to contain 1.
- For equidivided block matrix B, is it possible to make good approxymation to compute ρ by informations from its blocks, such as the sum of entries divided by dimension?
- Compute $E[\rho(\delta^{(k)}(D))]$ for random binary matrix D. Is it possible to find the distribution of $\rho(\delta^{(k)}(\bullet))$? Is there exists some meaningful limiting distribution? How about for the case of $\widehat{\delta}^{(k)}$?

About generalization

- How theses theory changes for the following possible generalizations?
  - restriction on reverse direction
  - restriction on both sub-walk and sub-co-walk
  - closed vertex sequences
  - colored arcs and restriction on the length of monotone sub-walks
- Is it possible to find some general formula for $\theta_m^{(k)}(D)$ based on binomial coefficients or inclusion-exclusion principle?