Fractional discrete Helly for pairs in a family of boxes

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joint work with
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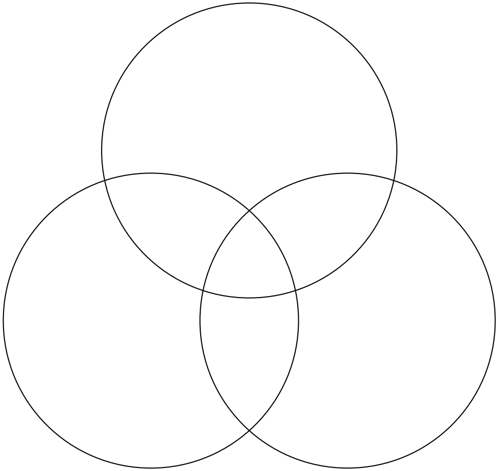
Helly's theorem

Helly's theorem for convex sets (1923)

For convex sets $X_1, \cdots .X_n \subseteq \mathbb{R}^d$, if every $d+1$ of them has nonempty intersection, then so is the total collection.



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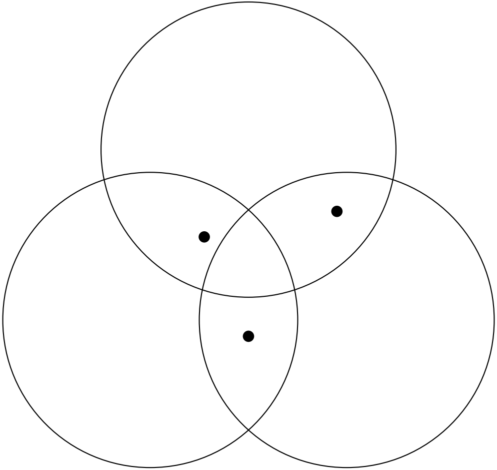

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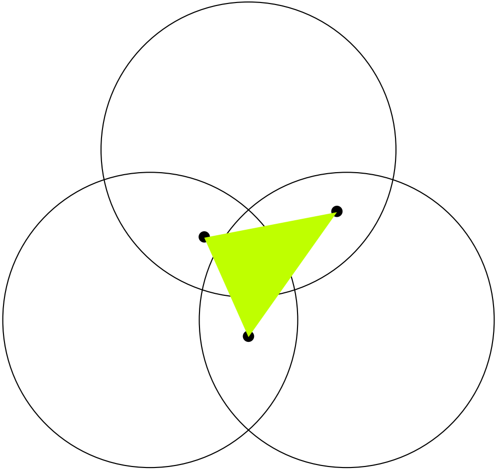


Helly-type theorems

Helly theorem for axis-parallel boxes

For axis-parallel boxes $B_1, \cdots B_n \subseteq \mathbb{R}^d$, if every pair intersects, then $\bigcap_i B_i \neq \emptyset$.

Fractional Helly theorem (Katchalski and Liu, 1979)

For any $\alpha \in (0,1)$, there exists $\beta = \beta(\alpha, d) > 0$ such that, for any convex sets $X_1, \cdots, X_n \subseteq \mathbb{R}^d$, if $\alpha\binom{n}{d+1}$ of $(d+1)$-tuples of them intersect, then there exists an intersecting subfamily of size at least βn.

S-Helly theorem (discrete Helly theorem) for axis-parallel boxes (Halman, 2008)

Let $S \subseteq \mathbb{R}^d$ be a set of points. For axis-parallel boxes $B_1, \cdots, B_n \subseteq \mathbb{R}^d$, if every $2d$ of them intersects at S, then so is the total collection. i.e., $S \cap \bigcap_i B_i \neq \emptyset$.

Fractional Helly theorems for axis-parallel boxes

Eckhoff, 1988

For axis-parallel boxes $B_1, \cdots, B_n \subseteq \mathbb{R}^d$ and a real number $\alpha \in (1 - \frac{1}{d}, 1]$, if at least $\alpha\binom{n}{2}$ pairs intersect, then there exists an intersecting subfamily of size at least $\left(1 - \sqrt{d(1-\alpha)}\right) n$.

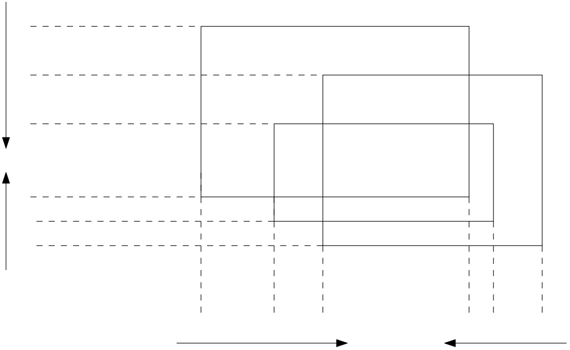
Edwards and Soberón, 2024, *preprint*

For any $\alpha \in (0, 1)$, there exists $\beta = \beta(\alpha, d)$ such that, for any point set $S \subseteq \mathbb{R}^d$ and axis-parallel boxes $B_1, \cdots, B_n \subseteq \mathbb{R}^d$, if $\alpha\binom{n}{d+1}$ of $(d+1)$-tuples intersects at S, then there exists a subfamily of size at least βn intersects at S.

Fractional Helly theorems for axis-parallel boxes

E., Kim and Lee, 2025, *preprint*

- For any d, there exists $N = N(d)$ such that, for any point set $S \subseteq \mathbb{R}^d$ and axis-parallel boxes $B_1, \cdots, B_N \subseteq \mathbb{R}^d$, if every pair of boxes intersects at S, then there is a subfamily of size $d+1$ intersects at S.
- For any d, there exists a constant $c_d \in (0,1)$ and a function $\beta_d : (c_d, 1] \to (0,1]$ such that the following holds: for any $\alpha \in (c_d, 1]$, point set $S \subseteq \mathbb{R}^d$, and axis-parallel boxes $B_1, \cdots, B_n \subseteq \mathbb{R}^d$, if at least $\alpha\binom{n}{2}$ pairs of boxes intersect at S, then there exists a subfamily of size at least $\beta_d(\alpha)n$ intersects at S.



For n pairwise S-intersecting axis-parallel boxes in $\mathbb{R}^d$, we have $2d$ permutations in S_n.

Permutation puzzle

- If $\cdots i \cdots j \cdots$ appears in every $2d$ permutations, it means the box i contains the box j.
- For fixed $j_1, \cdots, j_m$, if each permutation is a form in $\cdots i \cdots j_s \cdots$ for some s, then $B_{j_1} \cap B_{j_2} \cap \cdots \cap B_{j_m} \subseteq B_i$.
- Let $m_\sigma(s,t) = \max\{\sigma^{-1}(s), \sigma^{-1}(t)\}$. If there is s, t such that $\bigcap_\sigma \{\sigma(1), \sigma(2), \cdots, \sigma(m_\sigma(s,t))\} = \{i_1, \cdots, i_m, s, t\}$, then $B_s \cap B_t \subseteq B_{i_1} \cap \cdots \cap B_{i_m}$.
- For pairwise S-intersecting boxes, we can find S-intersecting $\max_{s,t} \left|\bigcap_\sigma \{\sigma(1), \cdots, \sigma\left(m_\sigma(s,t)\right)\}\right|$-tuple.

Question

If n is sufficient large, then

$$\max_{s,t} \left| \bigcap_{\sigma} \{\sigma(1), \sigma(2), \cdots, \sigma(m_\sigma(s,t))\} \right| \geq d+1$$

always?

Permutation Puzzle

Answer

For a, b, there exists $n = n(a, b)$ such that for every $A \subseteq S_n$ with $|A| \le a$, we have

$$\max_{s,t} \left| \bigcap_{\sigma \in A} \{\sigma(1), \sigma(2), \cdots, \sigma(m_\sigma(s,t))\} \right| \ge b + 1$$

Pattern graph

Let $A = \{\sigma_0, \sigma_1, \cdots, \sigma_{a-1}\} \subseteq S_n$.

- For $i, j \in [n]$ with $\sigma_0^{-1}(i) < \sigma_0^{-1}(j)$, A-pattern of i, j is $[i, j]_A = \{\sigma \in A \setminus \{\sigma_0\} \mid \sigma^{-1}(i) < \sigma^{-1}(j)\}$.
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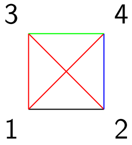

For permutation 1234, 1423, 3412

Pattern graph

Properties of pattern graph

- Monochromatic chain

$$\sigma_0^{-1}(s) < \sigma_0^{-1}(i_1) < \cdots < \sigma_0^{-1}(i_m) < \sigma_0^{-1}(t)$$

is chain for any other permutations in A. In other words,

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$\Rightarrow$ We can apply the multicolor Ramsey theorem.

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$$N(d) \leq n(2d, d) \leq R(\underbrace{d+1, d+1, \cdots, d+1}_{2^{2d-1}})$$

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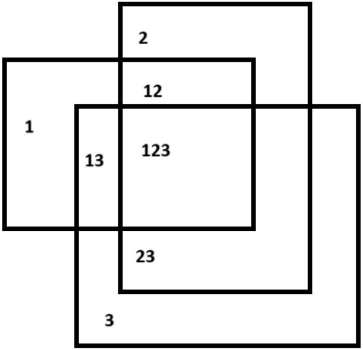
Question

What is the real asymptotic of $N(d)$?

Proof of $N(2) = 5$

In summary, the proof is based on the codeword.



Proof of $N(2) = 5$

- Codeword of axis-parallel boxes can be fully understood by codewords from its projections to each axis.
- Each projected codeword consists of one increasing chain and one decreasing chain.
- As a result, if $N(2) > 5$, then each size 2 code $\{1,2\}, \{1,3\}, \cdots, \{4,5\}$ must be contained in one of the size 3 set from each chain.
- By rearranging indices, the only case that makes it possible is $\{\{1,2,4\}, \{2,4,5\}, \{1,3,5\}, \{2,3,4\}\}$.
- Even for this case, any choice of S makes every pair of box intersects at S will have a triple intersects at S.

d-tuple intersecting pattern

E., Kim and Lee, 2025, *preprint*

- For any d, there exists $N = N(d)$ such that, for any point set $S \subseteq \mathbb{R}^d$ and axis-parallel boxes $B_1, \cdots, B_N \subseteq \mathbb{R}^d$, if every pair of boxes intersects at S, then there is a subfamily of size $d+1$ intersects at S.

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- There exists a constant C such that, for any point set $S \subseteq \mathbb{R}^d$ and axis-parallel boxes $B_1, \cdots, B_{Cd} \subseteq \mathbb{R}^d$, if every d-tuple of boxes intersects at S, then there is a subfamily of size $d+1$ intersects at S.

The end.