Computing the density of tautologies

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Propositional logic system

- Well-formed formulae(Syntax)
- Valuations(Semantics)
- Proofs(Syntactic)

Well-formed formulae(Syntax)

- Variables

  $x_0,\ x_1,\ \cdots,\ \overline{x_0},\ \overline{x_1},\ \cdots,\ \bot,\ \top$

  $x,\ x1,\ x11,\ x111,\ \cdots$

- Logical operators (Connectives)

  $\neg,\ \rightarrow,\ \vee,\ \wedge,\ |,\ \cdots$

- Grammar

  $\langle WFF\rangle\ ::=\ \langle Var\rangle\ |\ [\neg\langle WFF\rangle]\ |\ [\langle WFF\rangle\rightarrow\langle WFF\rangle]\ |\ \cdots$

  $\langle WFF\rangle\ ::=\ \langle Var\rangle\ |\ \neg\langle WFF\rangle\ |\ \rightarrow\langle WFF\rangle\langle WFF\rangle\ |\ \cdots$

- Length

  $\ell(x_i) = 1,\ \ell(\neg\phi) = \ell(\phi) + 1,\ \ell(\phi \rightarrow \psi) = \ell(\phi) + \ell(\psi) + 1,\ \cdots$

- Reduced length for the case without unary operator

  $\ell_2(x_i) = 1,\ \ell_2(\phi \rightarrow \psi) = \ell_2(\phi) + \ell_2(\psi),\ \cdots$

Well-formed formulae(Syntax)

- Variables

  $x_0, x_1, \cdots, \overline{x_0}, \overline{x_1}, \cdots, \bot, \top$

  $x, x1, x11, x111, \cdots$

- Logical operators (Connectives)

  $\neg, \rightarrow, \vee, \wedge, |, \cdots$

- Grammar

  $\langle WFF \rangle ::= \langle Var \rangle \mid [\neg \langle WFF \rangle] \mid [\langle WFF \rangle \rightarrow \langle WFF \rangle] \mid \cdots$

  $\langle WFF \rangle ::= \langle Var \rangle \mid \neg \langle WFF \rangle \mid \rightarrow \langle WFF \rangle \langle WFF \rangle \mid \cdots$

- Length

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- Reduced length for the case without unary operator

  $\ell_2(x_i) = 1,\ \ell_2(\phi \rightarrow \psi) = \ell_2(\phi) + \ell_2(\psi),\ \cdots$

Example

For the case of 3 variables, $\neg$, and $\rightarrow$ with length 5.

$$\neg\neg\neg\neg X \text{ type}: \ 3$$

$$\neg\neg[X \rightarrow Y] \text{ type}: \ 9$$

$$\neg[\neg X \rightarrow Y] \text{ type}: \ 9$$

$$\neg[X \rightarrow \neg Y] \text{ type}: \ 9$$

$$[\neg X \rightarrow \neg Y] \text{ type}: \ 9$$

$$[\neg\neg X \rightarrow Y] \text{ type}: \ 9$$

$$[X \rightarrow \neg\neg Y] \text{ type}: \ 9$$

$$X \rightarrow [Y \rightarrow Z] \text{ type}: \ 27$$

$$[X \rightarrow Y] \rightarrow Z \text{ type}: \ 27$$

So the total number of well-formed formulae is 111.

Valuations(Semantics)

- Truth table

<table>
<tr><th>ϕ</th><th>ψ</th><th>$\bot$</th><th>$\top$</th><th>$\neg\phi$</th><th>$\phi \rightarrow \psi$</th><th>$\phi \vee \psi$</th><th>$\phi \wedge \psi$</th><th>$\phi \mid \psi$</th><th>$\cdots$</th></tr>
<tr><td rowspan="2">F</td><td>F</td><td rowspan="4">F</td><td rowspan="4">T</td><td rowspan="2">T</td><td>T</td><td>F</td><td>F</td><td>T</td><td></td></tr>
<tr><td>T</td><td>T</td><td>T</td><td>F</td><td>T</td><td></td></tr>
<tr><td rowspan="2">T</td><td>F</td><td rowspan="2">F</td><td>F</td><td>T</td><td>F</td><td>T</td><td></td></tr>
<tr><td>T</td><td>T</td><td>T</td><td>T</td><td>F</td><td></td></tr>
</table>

- Semantic consequence : $A \vDash \phi$
- Tautology : $\vDash \phi$

In the propositional logic, a truth assignment on variables determines the valuation of well-formed formulae.

Example

For the case of 3 variables, $\neg$, and $\to$ with length 5.

$\neg\neg\neg\neg X$ type

$\neg\neg[X \to X]$ type : 3

$\neg[\neg X \to Y]$ type

$\neg[X \to \neg Y]$ type

$[\neg X \to \neg X]$ type : 3

$[\neg\neg X \to X]$ type : 3

$[X \to \neg\neg X]$ type : 3

$X \to [Y \to X]$ type : 9

$X \to [Y \to Y]$ type : 9

$X \to [X \to X]$ type : 3

$[X \to Y] \to Z$ type

So the total number of tautologies is $4 \cdot 3 + 2 \cdot 9 - 3 = 27$.

Proofs(Syntactic)

- Axiom schemes

  $K : \phi \to [\psi \to \phi]$

  $S : [\phi \to [\psi \to \eta]] \to [[\phi \to \psi] \to [\phi \to \eta]]$

  $\dots$
- Inference rules

  Modus ponens : $\phi,\ \phi \to \psi \Rightarrow \psi$

  Law of excluded middle : $\phi \vee \neg\phi$

  $\dots$
- Syntactic consequence : $A \vdash \phi$
- Deduction theorem : $H \vdash \phi \to \psi \Leftrightarrow H \cup \{\phi\} \vdash \psi$
- Theorem : $\text{Axioms} \vdash \phi$

Soundness and Completeness

- Soundness : Every theorem is a tautology.

  Most logic systems satisfy.
- Completeness : Every tautology is a theorem.

  Not generally true.

Complete Hilbert deduction system

- Variables : $x_0, x_1, \cdots$
- Connectives : $\neg, \rightarrow$
- Axioms schemes :

  $\phi \rightarrow [\psi \rightarrow \phi]$

  $[\phi \rightarrow [\psi \rightarrow \eta]] \rightarrow [[\phi \rightarrow \psi] \rightarrow [\phi \rightarrow \eta]]$

  $[\neg\psi \rightarrow \neg\phi] \rightarrow [\phi \rightarrow \psi]$
- Inference rules : Modus ponens ($\phi, \phi \rightarrow \psi \vdash \psi$)

Why the density of tautologies?

Density

$$\text{The density} = \lim_{n \to \infty} \text{The probability among length } n$$

Theorem-side : Automated theorem proving

Probability analysis for the random theorem generation

Tautology-side : Satisfiability problem

$$\phi \text{ is satisfiable} \Leftrightarrow \neg\phi \text{ is not a tautology}$$

Propositional logic systems with one variable

Connectives	The density
$\{\neg, \to\}$[1]	$\frac{1}{4\sqrt{13}} + \frac{1}{4\sqrt{17}} + \frac{1}{2\sqrt{2(\sqrt{221}-9)}} + \frac{15}{2\sqrt{442(\sqrt{221}-9)}}$
$\{\mid\}$[2]	$\frac{3\sqrt{2}+2\sqrt{3}-6}{6\sqrt{7-2\sqrt{3}-2\sqrt{2}}} \approx 0.33819$
$\{\neg, \wedge\}$[2]	$\frac{12-3\sqrt{2}-2\sqrt{3}}{24\sqrt{4-\sqrt{3}-\sqrt{2}}} \approx 0.19360$
$\{\mathrm{NOR}\}$[2]	≈ 0.05373
$\{\neg, \vee\}$[2]	≈ 0.55138
$\{\neg, \wedge, \vee\}$[2]	≈ 0.26081
$\{\neg, \wedge, \to\}$[2]	≈ 0.36305
$\{\neg, \wedge, \to, \leftrightarrow\}$[2]	≈ 0.33729

[1] On the asymptotic density of tautologies in logic of implication and negation (M. Zaionc, 2005)

[2] Density of tautologies in logics with one variable (L. Aszalós and T. Herendi, 2012)

Propositional logic systems with one variable

Connectives	The density
$\{\neg, \to\}$[1]	≈ 0.42324
$\{\mid\}$[2]	$\frac{3\sqrt{2}+2\sqrt{3}-6}{6\sqrt{7-2\sqrt{3}-2\sqrt{2}}} \approx 0.33819$
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Propositional logic systems without unary operators

- On the density and the structure of the Peirce-like formulae (A. Genitrini, J. Kozik, and G. Matecki, 2008)
- Tautologies over implication with negative literals (H. Fournier, D. Gardy, A. Genitrini, and M. Zaionc, 2010)
- 2-Xor revisited: satisfiability and probabilities of functions (É. de Panafieu, D. Gardy, B. Gittenberger, and M. Kuba, 2016)

...

Contains Motzkin structure

- On the number of unary-binary tree-like structures with restrictions on the unary height (O. Bodini, D. Gardy, B. Gittenberger, and Z. Gołębiewski, 2018)
- Unary profile of lambda terms with restricted De Bruijn indices (K. Grygiel and I. Larcher, 2021)

...

A method to compute the exact density of tautologies in propositional logic systems with m variables

- Construct a system of equations for generating functions.
- Define well-organized sets to solve the system by divide and conquer method.
- Provide a way to compute the exact value.

Practically better method to verify the exact value of the density

- Define an s-cut operator and s-cut solution.
- Define a shifted s-cut operator to use the iterative method.
- Provide a computed result to show the effect of memory-time tradeoff.

Asymptotic behaviors of the density of tautologies as the number of variables goes to the infinity

- Introduce the definition of simple tautologies.
- Define strong and weak category for well-formed formulae.
- Compute asymptotic lower bounds and an upper bound of the density of tautologies.

Existence by Drmota-Lalley-Woods theorem

The number of length n tautologies is approximately

$$\frac{1}{\rho^n}\sum_{k\geq 0}\frac{d_k}{n^{k+\frac{3}{2}}}$$

for some computable constants d_k's and ρ.

Computation of the value by Szegő lemma

For two generating functions $U(z)$, $V(z)$ with the common nearest simple singularity ρ around 0,

$$\lim_{n\to\infty}\frac{[z^n]U(z)}{[z^n]V(z)}=\lim_{z\to\rho^-}\frac{\frac{U(z)-U(\rho)}{\sqrt{1-z/\rho}}}{\frac{V(z)-V(\rho)}{\sqrt{1-z/\rho}}}=\lim_{z\to\rho^-}\frac{-2\rho U'(z)\sqrt{1-z/\rho}}{-2\rho V'(z)\sqrt{1-z/\rho}}.$$

Generating function of well-formed formulae with m variables

The generating function $W(z)$ is defined as

$$W(z) = \sum_{\phi \in \langle WFF \rangle} z^{\ell(\phi)}$$

Then, in the recursive structure

$$\langle WFF \rangle ::= \langle Var \rangle \mid [\neg \langle WFF \rangle] \mid [\langle WFF \rangle \rightarrow \langle WFF \rangle],$$

- $\langle Var \rangle$ corresponds to mz^1
- $[\neg \langle WFF \rangle]$ corresponds to $zW(z)$
- $[\langle WFF \rangle \rightarrow \langle WFF \rangle]$ corresponds to $W(z)zW(z) = zW(z)^2$

Thus,

$$W(z) = mz + zW(z) + zW(z)^2$$

If $m = 3$, then $W(z) = 3z + 3z^2 + 12z^3 + 30z^4 + 111z^5 + \cdots$.

Falsity set

Define the falsity set F_ϕ as the set of valuations make ϕ false.

$$F_\phi = \{v \in \mathcal{VA} \mid v(\phi) = [\![\phi; v]\!] = \mathrm{False} = 0\}$$

- The number of valuations is 2^m.
- The number of possible falsity sets is 2^{2^m}.
- $F_{\neg\phi} = F_\phi^c$
- $F_{\phi\to\psi} = F_\psi \setminus F_\phi$
- $F_{\phi\vee\psi} = F_\phi \cap F_\psi$
- $F_{\phi\wedge\psi} = F_\phi \cup F_\psi$
- ϕ is a tautology $\Leftrightarrow F_\phi = \emptyset$.
- $W_A(z) := \sum_{F_\phi = A} z^{\ell(\phi)}$

System of equations

- If $A = F_{x_i}$ for a variable x_i,

$$W_A(z) = z + zW_{A^c}(z) + \sum_{C \setminus B = A} zW_B(z)W_C(z).$$

- Otherwise,

$$W_A(z) = zW_{A^c}(z) + \sum_{C \setminus B = A} zW_B(z)W_C(z).$$

Note that a system of quadratic equations is not always solvable by radicals. For example,

$$\begin{cases} x = z^2 \\ y = x^2 \\ z = yz + \rho \end{cases}$$

is equivalent to $z^5 - z + \rho = 0$.

Well-organized partition

For sets of valuations $A, B \subseteq \mathcal{VA}$, define

$$\mathcal{I}_{A;B} = \{Y \mid A \setminus B \subseteq Y \subseteq A \cup B\}.$$

- $\mathcal{I}_{;B} = \{\mathcal{I}_{A;B} \mid A \cap B = \emptyset\}$ is the partition of the power set of valuations described by equivalence relation

$$Y \sim Z \Leftrightarrow Y \setminus B = Z \setminus B \Leftrightarrow Y \cup B = Z \cup B \Leftrightarrow Y \Delta Z \subseteq B.$$

- $\mathcal{I}_{A^c;B} = \{Y^c \mid Y \in \mathcal{I}_{A;B}\}$
- $\mathcal{I}_{A \setminus C;B} = \{Y \setminus Z \mid Y \in \mathcal{I}_{A;B}, Z \in \mathcal{I}_{C;B}\}$
- $\mathcal{I}_{A \cap C;B} = \{Y \cap Z \mid Y \in \mathcal{I}_{A;B}, Z \in \mathcal{I}_{C;B}\}$
- $\mathcal{I}_{A \cup C;B} = \{Y \cup Z \mid Y \in \mathcal{I}_{A;B}, Z \in \mathcal{I}_{C;B}\}$

Well-organized partition

- $\mathcal{I}_{;B}$ is isomorphic to $\mathcal{P}(B^c)$ as a poset.
- $\mathcal{I}_{A;B}$ is a translation of $\mathcal{P}(B)$ as a poset.
- $\mathcal{I}_{A;B}$ is a set-operational coset.
- $\mathcal{I}_{\emptyset;B}$ is an order ideal.
- $\mathcal{I}_{-;B} := \mathcal{I}_{B^c;B}$ is a filter.
- $\mathcal{I}_{A;B\cup\{y\}} = \mathcal{I}_{A;B} \cup \mathcal{I}_{A\cup\{y\};B}$ for $y \notin A \cup B$.

If $I_{A;B}(z) := \sum_{Y\in\mathcal{I}_{A;B}} W_Y(z)$,

$$I_{A;B}(z) = (\#F_{x_i} \in \mathcal{I}_{A;B})z + zI_{A^c;B}(z) + \sum_{\substack{C\setminus D=A\setminus B \\ C\cap B=D\cap B=\emptyset}} zI_{C;B}(z)I_{D;B}(z).$$

Linear dependencies

- $I_{A;B}(z)$ is a linear combination of

$$\{I_{\emptyset;B}(z)\} \cup \{I_{C;B'}(z) \mid C \subsetneq A \setminus B,\ C \cap B' = \emptyset,\ |B'| = |B|+1,\ B \subseteq B'\}.$$

Precisely,

$$I_{A;B}(z) = (-1)^{|A\setminus B|} I_{\emptyset;B}(z) + \sum c_{C;B'} I_{C;B'}(z).$$

- $I_{A;B}(z)$ is a linear combination of

$$\{I_{-;B}(z)\} \cup \{I_{C;B'}(z) \mid A \setminus B \subseteq C,\ C \cap B' = \emptyset,\ |B'| = |B|+1,\ B \subseteq B'\},$$

Precisely,

$$I_{A;B}(z) = (-1)^{|B^c\setminus A|} I_{-;B}(z) + \sum c_{C;B'} I_{C;B'}(z).$$

Linear dependencies

$I_{A';B}(z)$ is a linear combination of

$$\{I_{A;B}\} \cup \{I_{C;B'} \mid |B'| = |B| + 1,\ B \subseteq B'\}.$$

Precisely,

$$I_{A';B}(z) = \pm I_{A;B}(z) + \sum c_{C;B'} I_{C;B'}(z).$$

Basis

For disjoint A, B,

$$I_{A;B}(z) = (-1)^{|A|+|B|} \sum_{B \subseteq B' \subseteq A \cup B} (-1)^{|B'|} I_{\emptyset;B'}(z),$$

$$I_{A;B}(z) = (-1)^{|A|} \sum_{B \subseteq B' \subseteq A^c} (-1)^{|B'|} I_{-;B'}(z).$$

Solvability of the system of equations for well-organized partitions

$$I_{A;B}(z) = (\#F_{x_i} \in \mathcal{I}_{A;B})z + zI_{A^c;B}(z) + \sum_{\substack{C \setminus D = A \setminus B \\ C \cap B = D \cap B = \emptyset}} zI_{C;B}(z)I_{D;B}(z)$$

is a nontrivial, at-most-quadratic equation of $I_{A;B}(z)$. Hence, we can solve if for every pair of (A, B) from large B to small B.

$$I_{-;B}(z) = (\#F_{x_i} \in \mathcal{I}_{-;B})z + zI_{\emptyset;B}(z) + zI_{\emptyset;B}(z)I_{-;B}(z)$$
$$I_{\emptyset;B}(z) = \sum_{B \subseteq B'} (-1)^{|B'|} I_{-;B'}(z) = (-1)^{|B|} I_{-;B}(z) + \sum_{B \subsetneq B'} (-1)^{|B'|} I_{-;B'}(z)$$

Definitions

- $m_{-;B} := (\#F_{x_i} \in \mathcal{I}_{-;B})$,
- $\sigma_B := (-1)^{|B|}$,
- $I_B^{\uparrow}(z) := \sum_{B \subsetneq B'} \sigma_{B'} I_{-;B'}(z)$,

$$I_{-;B}(z) = m_{-;B}z + zI_{\emptyset;B}(z) + zI_{\emptyset;B}(z)I_{-;B}(z)$$
$$I_{\emptyset;B}(z) = \sigma_B I_{-;B}(z) + I_B^{\uparrow}(z)$$

$$I_{-;B}(z) = z(m_{-;B} + I_B^{\uparrow}(z)) + z(\sigma_B + I_B^{\uparrow}(z))I_{-;B}(z) + z\sigma_B I_{-;B}(z)^2$$

Here, the discriminant is

$$D_B(z) := (1 - (\sigma_B + I_B^{\uparrow}(z))z)^2 - 4\sigma_B z^2(m_{-;B} + I_B^{\uparrow}(z)).$$

Szegő's lemma

$$\lim_{n\to\infty} \frac{[z^n]I_{A;B}(z)}{[z^n]W(z)} = \lim_{z\to\rho^-} \frac{2\rho I'_{A;B}(z)\sqrt{1 - z/\rho}}{2\rho W'(z)\sqrt{1 - z/\rho}}.$$

where $W(z)$ has the nearest simple singularity at $\frac{1}{2\sqrt{m}+1}$

Definitions

- $\rho_0 := \frac{1}{2\sqrt{m}+1}$,
- $\alpha_B := I_{-;B}(\rho_0)$,
- $\alpha_B^{\uparrow} := I_B^{\uparrow}(\rho_0) = \sum_{B \subsetneq B'} \sigma_{B'} \alpha_{B'}$,
- $\beta_B := 2\rho_0 \lim_{z \to \rho_0^-} I'_{-;B}(z) \sqrt{1 - \frac{z}{\rho_0}}$,
- $\beta_B^{\uparrow} := \sum_{B \subsetneq B'} \sigma_{B'} \beta_{B'}$,
- $d_B := \frac{D_B(\rho_0)}{\rho_0^2} = (\frac{1}{\rho_0} - \sigma_B - \alpha_B^{\uparrow})^2 - 4\sigma_B(m_{-;B} + \alpha_B^{\uparrow})$.

Goal

$$\lim_{n\to\infty} \frac{[z^n] I_{A;B}(z)}{[z^n] W(z)} = \frac{\sigma_{A \setminus B} \sum_{B \subseteq B' \subseteq (A\setminus B)^c} \sigma_{B'} \beta_{B'}}{\beta_{\mathcal{V}\mathcal{A}}}$$

Initial conditions

$$\alpha_{\mathcal{VA}} = \sqrt{m}$$
$$\beta_{\mathcal{VA}} = \sqrt{2m + \sqrt{m}}$$

Recursive relations

$$\alpha_B = \frac{2\sqrt{m} + 1 - \sigma_B - \alpha_B^{\uparrow} - \sqrt{d_B}}{2\sigma_B}$$
$$\beta_B = \beta_B^{\uparrow} \frac{-1 + \frac{2\sqrt{m}+1+\sigma_B-\alpha_B^{\uparrow}}{\sqrt{d_B}}}{2\sigma_B}$$

Here, to write a program, we may use the binary representations. Since $B \subseteq B'$ implies their corresponding integers satisfy $b \leq b'$, we can write a code with simple for-loops for those recursive relations.

The density of tautologies with m variables, $\neg$ and $\rightarrow$

$$\lim_{n\to\infty} \frac{[z^n]I_{\emptyset;\emptyset}(z)}{[z^n]W(z)} = \frac{\sum_{B\subseteq \mathcal{V}\mathcal{A}} \sigma_B \beta_B}{\sqrt{2m+\sqrt{m}}}$$

$m = 1$	0.42324..
$m = 2$	0.33213..
$m = 3$	0.27003..
$m = 4$	0.22561..

Since every β_B is a constructible number, so is the density.

Converging speed

For a system of quadratic equations for generating functions, nth coefficients are in

$$\frac{1}{\rho^n}\sum_{k\geq 0}\frac{d_k}{n^{k+\frac{3}{2}}}$$

form, so

$$\left|\frac{[z^n]A(z)}{[z^n]B(z)} - \lim_{n\to\infty}\frac{[z^n]A(z)}{[z^n]B(z)}\right| \simeq O(\frac{1}{n})$$

Memory usage

For a sysetm of quadratic equations for generating functions A_1, $\cdots$ A_N, to compute $[z^n]A_i(z)$, we need all of values $[z^s]A_j(z)$ for $0 \leq s < n$ and every j's.

Is there any way to use same amount of informations to compute the better estimation than $\frac{[z^n]A(z)}{[z^n]B(z)}$?

A basic generating function

Set a generating function $Y(z)$ satisfies

- $Y(z) = f(z) + g(z)Y(z) + h(z)Y(z)^2$,
- $\deg g, \deg h < \infty$,
- $\frac{Y_{n+1}}{Y_n} \to \frac{1}{\rho} > 1$,
- $\frac{f_n}{Y_n} \to \gamma$.

$$\sum_{u=0}^{\deg h} \sum_{v=s+1}^{n-u-s-1} h_u \frac{Y_v Y_{n-u-v}}{Y_n} \simeq 1 - \gamma - g(\rho) - 2h(\rho) \sum_{k=0}^{s} Y_k \rho^k =: \zeta_s$$

A system of quadratic equations for generating functions

Set generating functions $A_1(z), \cdots, A_N(z)$ satisfy

- $\frac{A_{in}}{Y_n} \to \beta_i$,
- $A_i(z) = f_i(z) + \sum_j g_{ij}(z)A_j(z) + \sum_{j,k} h_{ijk}(z)A_j(z)A_k(z)$,
- $\deg g_{ij}, \deg h_{ijk} < \infty,\ h | h_{ijk}$,
- $\frac{f_{in}}{Y_n} \to \gamma_i$.

$$\begin{aligned}\beta_i \simeq \gamma_i &+ \sum_j g_{ij}(\rho)\beta_j \\ &+ \sum_{j,k} h_{ijk}(\rho)(A_j^{\le s}(\rho)\beta_k + A_k^{\le s}(\rho)\beta_j) \\ &+ \sum_{j,k} \zeta_s \frac{h_{ijk}}{h}(\rho)\beta_j\beta_k\end{aligned}$$

s-cut operator and s-cut solution

Define the s-cut operator $C_s(x_1, \cdots, x_N) = (c_1, \cdots, c_N)$ as

$$
\begin{aligned}
c_i = \gamma_i &+ \sum_j g_{ij}(\rho) x_j \\
&+ \sum_{j,k} h_{ijk}(\rho)(A_j^{\leq s}(\rho) x_k + A_k^{\leq s}(\rho) x_j) \\
&+ \sum_{j,k} \zeta_s \frac{h_{ijk}}{h}(\rho) x_j x_k
\end{aligned}
$$

and a fixed point $(\beta_1^{(s)}, \cdots, \beta_N^{(s)})$ of C_s as an s-cut solution.

Basic properties

With proper conditions,

- $C_s(\beta_1, \cdots, \beta_N) \to (\beta_1, \cdots, \beta_N)$ as $s \to \infty$,
- $\lim_{s\to\infty} \zeta_s = 0$,
- $\beta_i = \gamma_i + \sum_j g_{ij}(\rho)\beta_j + \sum_{j,k} h_{ijk}(\rho)(A_j(\rho)\beta_k + A_k(\rho)\beta_j)$.

This give a relation between generating function values at the singularity and ratios.

Natural partition

- $Y(z) = A_1(z) + \cdots + A_N(z)$, $1 = \beta_1 + \cdots + \beta_N$
- $f(z) = f_1(z) + \cdots + f_N(z)$
- $g(z) = g_{1j}(z) + \cdots + g_{Nj}(z)$ for every j
- $2h(z) = h_{1jk}(z) + h_{1kj}(z) + \cdots + h_{Njk}(z) + h_{Nkj}(z)$ for every j, k

For example, $W(z) = \sum_{A \subset \mathcal{VA} \setminus B} I_{A;B}(z)$ form a natural partition.

Shifted s-cut operator

Since the Jacobian matrix of C_s can have larger norm than 1, we may consider a shifted

$$\widetilde{C_s^\sigma}(x) := C_s(x) - \sigma \left(\sum_{i=1}^{N} x_i - 1 \right) \cdot (1, 1, \cdots, 1)$$

to use the iteration method.

Convergence of s-cut solutions

For a natural partition system, if we have proper contraction factor K after shifting,

$$|\beta - \beta^{(s)}| \leq \frac{1}{1-K}|\beta - C_s(\beta)| \to 0$$

as $s \to \infty$.

Practically, for our density of tautologies with one variable, $\neg$ and $\to$,

	ratio at s	s-cut
$s = 10$	0.3101796...	0.4242620...
$s = 100$	0.4187317...	0.4232740...
$s = 1000$	0.4227880...	0.4232396...
$s = 10000$	0.4231935...	0.4232386...

s-cut solutions show better convergence result to the real limit value 0.4232385....

Estimator ν

For a well-formed formula ϕ, $\nu(\phi)$ is

$$(\#\text{distinct variables used in } \phi) - \frac{\ell(\phi)}{2}$$

This estimator estimates the generating function value at the singularity.

- $\nu(\phi) \leq \frac{1}{2}$
- $\nu(\phi) \leq -\frac{1}{2}$ if ϕ is a tautology
- $\nu(\phi) \leq -1$ if ϕ is an antilogy
- The density of tautologies seems related to

$$\frac{m^{\nu(\langle Tau \rangle)}}{m^{\nu(\langle WFF \rangle)}} = \frac{m^{-\frac{1}{2}}}{m^{\frac{1}{2}}} = \frac{1}{m}$$

Simple tautologies

A tautology τ is a simple tautology if

$$\tau = \phi_1 \to [\phi_2 \to [\cdots \to [\phi_n \to p] \cdots]]$$

where $\phi_i = p$ for some variable p.
A tautology τ is a strict simple tautology if

$$\tau = p \to [\phi_2 \to [\cdots \to [\phi_n \to p] \cdots]]$$

where $\phi_2, \cdots, \phi_n \neq p$. Here, $p \to p$ is a strict simple tautology.

For a tautology ϕ, $\nu(\phi) = -\frac{1}{2}$ if and only if ϕ is a simple tautology that has no $\neg$ and p is the only variable used in ϕ more than once.

This definition is borrowed from *Probability distribution for simple tautologies* (M. Zaionc, 2006) and *Tautologies over implication with negative literals* (H. Fournier, D. Gardy, A. Genitrini, and M. Zaionc, 2010).

Lower bounds of the density

- Tautologies of the form $p \to [\phi \to p]$ and $\neg\neg\langle Tau \rangle$ give a lower bound

$$\frac{\sqrt{m}}{4(\sqrt{m}+1)(2\sqrt{m}+1)^2}$$

- Strict simple tautologies give a lower bound

$$\frac{m}{(2m+3\sqrt{m}+2)^2} = \frac{1}{4m} - \frac{3}{4m\sqrt{m}} + \frac{19}{16m^2} + O(\frac{1}{m^2\sqrt{m}})$$

- Simple tautologies give a lower bound

$$\frac{m(4m+6\sqrt{m}+3)}{(\sqrt{m}+1)^2(2m+3\sqrt{m}+2)^2} = \frac{1}{m} - \frac{7}{2m\sqrt{m}} + \frac{7}{m^2} + O(\frac{1}{m^2\sqrt{m}})$$

If we have a set of tautologies $\mathcal{B}$, then from these, we may recursively define induced tautologies by categorizing well-formed formulae into

- known-tautologies $\mathcal{T}$,
- known-antilogies $\mathcal{A}$,
- unknowns $\mathcal{U}$.

There exists two ways to define recursive structure: strong and weak.

For the strong category, we will use $\mathcal{B}_*$, $\mathcal{T}_*$, $\mathcal{A}_*$, $\mathcal{U}_*$, and for the weak category and for the weak category, we will use $\mathcal{B}^*$, $\mathcal{T}^*$, $\mathcal{A}^*$, $\mathcal{U}^*$.

Strong category

ϕ	$\mathcal{T}_*$	$\mathcal{A}_*$	$\mathcal{U}_*$	$\mathcal{T}_*$	$\mathcal{A}_*$	$\mathcal{U}_*$	$\mathcal{T}_*$	$\mathcal{A}_*$	$\mathcal{U}_*$
ψ		$\mathcal{T}_*$			$\mathcal{A}_*$			$\mathcal{U}_*$	
$\neg\psi$		$\mathcal{A}_*$			$\mathcal{T}_*$			$\mathcal{U}_*$	
$\phi \to \psi$		$\mathcal{T}_*$		$\mathcal{A}_*$	$\mathcal{U}_*$	$\mathcal{U}_*$		$\mathcal{U}_*$	

$$T_*(z) = B_*(z) + zA_*(z) + zW(z)T_*(z),$$
$$U_*(z) = mz - B_*(z) + zU_*(z) + z[W(z)U_*(z) + A_*(z)^2 + U_*(z)A_*(z)],$$
$$A_*(z) = zT_*(z) + zT_*(z)A_*(z).$$

Here, $A_* = \frac{zT_*}{1-zT_*}$, so by substituting this in $T_* = B_* + zA_* + zT_*W$, we can solve the equation easily since it gives a quadratic equation.

Weak category

ϕ	$\mathcal{T}^*$	$\mathcal{A}^*$	$\mathcal{U}^*$	$\mathcal{T}^*$	$\mathcal{A}^*$	$\mathcal{U}^*$	$\mathcal{T}^*$	$\mathcal{A}^*$	$\mathcal{U}^*$
ψ		$\mathcal{T}^*$			$\mathcal{A}^*$			$\mathcal{U}^*$	
$\neg\psi$		$\mathcal{A}^*$			$\mathcal{T}^*$			$\mathcal{U}^*$	
$\phi \to \psi$		$\mathcal{T}^*$		$\mathcal{A}^*$	$\mathcal{T}^*$	$\mathcal{U}^*$	$\mathcal{U}^*$	$\mathcal{T}^*$	$\mathcal{U}^*$

$$T^*(z) = B^*(z) + zA^*(z) + z[W(z)T^*(z) + A^*(z)W(z) - A^*(z)T^*(z)],$$
$$U^*(z) = mz - B^*(z) + zU^*(z) + zU^*(z)W(z),$$
$$A^*(z) = zT^*(z) + zT^*(z)A^*(z).$$

After we check the analyticity condition, we can find a series solution for

$$\lim_{n\to\infty} \frac{[z^n]T^*(z)}{[z^n]W(z)}$$

where the series base is $\frac{1}{\sqrt{m}}$.

Lower bounds of the density

- The strong-categorized tautologies from simple tautolgies give a lower bound

$$\frac{1}{m} - \frac{7}{2m\sqrt{m}} + \frac{31}{4m^2} + O(\frac{1}{m^2\sqrt{m}})$$

- The weak-categorized tautologies from simple tautolgies give a lower bound

$$\frac{1}{m} - \frac{5}{2m\sqrt{m}} + \frac{29}{8m^2} + O(\frac{1}{m^2\sqrt{m}})$$

- The weak-categorized tautologies from simple tautolgies include second kind simple tautology concept give a lower bound

$$\frac{1}{m} - \frac{7}{4m\sqrt{m}} + \frac{5}{4m^2} + O(\frac{1}{m^2\sqrt{m}})$$

Upper bound of the density

By counting false formulae, we have an upper bound of the density of tautologies as

$$\frac{1}{2}+\frac{\sqrt{m}}{2\sqrt{5m+6\sqrt{m}+4}}=\frac{\sqrt{5}+1}{2\sqrt{5}}-\frac{3}{10\sqrt{5m}}+\frac{7}{100\sqrt{5}m}+O(\frac{1}{m\sqrt{m}})$$

Asymptotic results

The density of tautologies satisfy

$$\frac{1}{m}+O(\frac{1}{m\sqrt{m}})\leq\bullet\leq\frac{\sqrt{5}+1}{2\sqrt{5}}+O(\frac{1}{\sqrt{m}})$$

where the lower bound is conjectured as tight.

The density of tautologies with m variables, $\neg$, and $\rightarrow$

$m = 1$	0.42324..
$m = 2$	0.33213..
$m = 3$	0.27003..
$m = 4$	0.22561..

Is it possible to compute the exact number for $m \geq 5$ practically?

The better estimation by s-cut for the density of tautologies with one variable, $\neg$ and $\rightarrow$

0.4232385...	ratio at s	s-cut
$s = 10$	0.3101796...	0.4242620...
$s = 100$	0.4187317...	0.4232740...
$s = 1000$	0.4227880...	0.4232396...
$s = 10000$	0.4231935...	0.4232386...

Is it possible to compute how s-cut solution is better quantitatively?

Asymptotic bounds for the density of tautologies as the number of variables $m \to \infty$

$$\frac{1}{m} + O(\frac{1}{m\sqrt{m}}) \leq \bullet \leq \frac{\sqrt{5}+1}{2\sqrt{5}} + O(\frac{1}{\sqrt{m}})$$

Is it possible to reduce the upper bound to $\frac{1}{m} + O(\frac{1}{m\sqrt{m}})$?

Thank you.